Chapter 8.1 - Matchings II

Let $S_1, S_2, \ldots, S_n$ be nonempty finite sets. Then this collection of sets has a **system of distinct representatives** if there exist n distinct elements $x_1, x_2, \ldots, x_n$ such that $x_i \in S_i$ for $1 \le i \le n$.

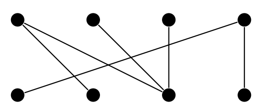
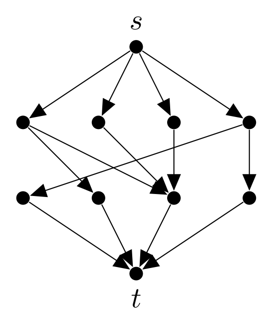
1: Find a system of distinct representatives for the following sets

$$S_1 = \{1,2,3\} S_2 = \{2,4,6\} S_3 = \{2,5,6\} S_4 = \{3,4,5\} S_5 = \{1,4,6\}$$

Theorem 8.4 (Original formulation of Hall's Theorem) A collection $\{S_1, S_2, \ldots, S_n\}$ of nonempty finite sets has a system of distinct representatives if and only if for each integer k with $1 \le k \le n$, the union of any k of these sets contains at least k elements.

2: Use Hall's theorem to prove Theorem 8.4.

How to (effectively) find a maximum matching in a bipartite graph?

Convert bipartite graph  into maximum flow problem 

by adding two new vertices s and t, joining s to one part of the bipartition and t to the other part. Assign capacity to every edge 1. (Generalizes also to edges that have costs.)

3: Why maximum matching corresponds to maximum flow correspond to each other?

Recall set of vertices is **independent** if they do not induce any edge. **Independence number** $\alpha(G)$ is the maximum cardinality of an independent set in G.

4: What is $\alpha(C_5)$, $\alpha(K_5)$ and independence number of Petersen's graph?

Vertex cover in a graph G is a set of vertices X such that every edge in G has an endpoint in X. **Vertex covering number** $\beta(G)$ is the minimum cardinality of a vertex cover in G.

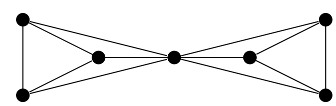
5: What is $\beta(C_5)$, $\beta(K_5)$ and vertex covering number of Petersen's graph?

Theorem 8.8 For every graph G of order n containing no isolated vertices,

$$\alpha(G) + \beta(G) = n$$

6: Prove Theorem 8.8

7: Determine the $\alpha(G), \beta(G), \alpha'(G), \beta'(G)$ for the following graph



8: *8.9* For each integer i with $1 \leq i \leq 4$, give an example of a connected graph G_i of smallest order such that $\alpha(G_i) + \alpha'(G_i) = 5$ and $\alpha(G_i) = i$

9: Let G be a bipartite graph with vertex sets V_1 and V_2. Let A be the set of vertices of maximal degree. Shot that there is a complete matching from $A \cap V_1$ into V_2.

10: Let $A_1, \ldots, A_n$ be sets, where each has $n-1$ elements and no two sets are equal. Is there a system of distinct representatives for $A_1, \ldots, A_n$?

11: Consider all subsets of cardinality $n-1$ of the set $\{1, 2, \ldots, n\}$. How many systems of distinct representatives this system has?

12: For $k \geq 2$, prove that for a k-dimensional hypercube Q_k has at least $2^{2^{k-2}}$ perfect matchings.