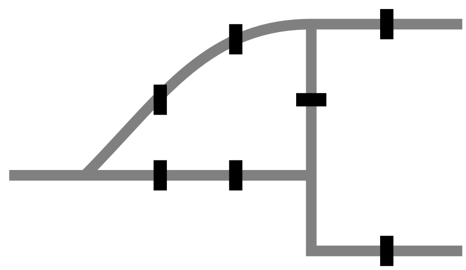
Chapter 6.1 - Eulerian Graphs

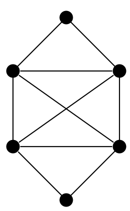
Historical problem: Take a walk in Königsberg and traverse every bridge exactly once. Bridges are black.



1: Is it possible to traverse every bridge exactly once?

Recall that trail is a sequence of vertices and edges without repeating edges. Circuit is a closed trial. A graph is **Eluerian** if it contains a circuit that contains all edges. Such circuit is called **Eulerian circuit**.

2: Find Eulerian circuit in the following graph



3: Decide if K_5 and the Petersen's graph are Eulerian.

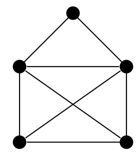
4: Show that if G is Eulerian, then degree of every vertex is even.

Theorem 6.1 A nontrivial connected graph G is Eulerian if and only if every vertex of G has even degree.

5: Show that if a connected graph G has every vertex of even degree, then G is Eulerian. (Hint: Take longest circuit)

Eulerian trail in a graph G is a trail in G containing all edges and does not start and end at the same vertex.

6: Find an Eulerian trail in the following graph



Corollary 6.2 A connected graph G contains an Eulerian trail if and only if exactly two vertices of G have odd degree. Furthermore, each Eulerian trail of G begins at one of these odd vertices and ends at the other.

7: Prove Corollary 6.2 using Theorem 6.1.

8: Does every Eulerian bipartite graph have an even number of edges? Explain.

9: Does every Eulerian simple graph with an even number of vertices have an even number of edges? Explain.

10: Prove or disprove the following statement: If G is a graph with edges e and f that share a common vertex v, then there is an Eulerian circuit which goes through the edge e and then immediately after through f.

11: Only one graph of order 5 has the property that the addition of any edge produces an Eulerian graph. What is it?

12: Notice that the Eulerian graph can be defined also for directed graphs. Show that a directed graph G is Eulerian if and only if the graph is connected and at each vertex the in-degree equals the out-degree.

13: Prove that if P and Q are paths of maximum length in a connected graph G, then P and Q have at least one vertex in common.

14: *Open problem* In a connected graph, pick any three paths of maximum length. Is there always a vertex that lies on all of them?

For next time read Chapter 6.2 - Hamilton Cycles.