

Chapter 9.2 - Graphs on Surfaces

Last time we saw that drawing graph in the plane is the same problem as drawing it in the sphere.

What about drawing on other surfaces?

Drawing on the torus is the same as drawing on the following where parallel edges with arrows are identified. (This how would you make a torus from a piece of paper.)



1: Draw K_5 and $K_{3,3}$ on torus.

Torus can be thought of as adding one handles to a sphere (or adding a bridge to the plane)

Other *famous* surfaces are the Projective plane and the Klein bottle.

Drawing on the Projective plane is equivalent to drawing on the Möbius strip - try on a real one.



Drawing on the Klein Bottle is like



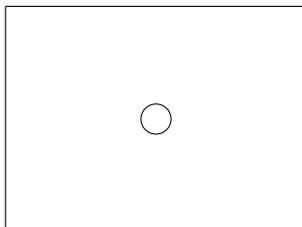
Projective plane and Klein Bottle are called non-orientable surface since left/right or up/down does not make any sense on these surfaces.

2: Draw K_5 and $K_{2,2,2}$ in the Projective Plane and in the Klein bottle.

An alternative description of the projective plane is to use plane and add a crosscap. A **crosscap** in the plane is point where edges may cross without actually crossing. It can be obtained by cutting a circle in a surface and identifying opposite points.

Also notice that edges going through cross cap **MUST** cross. Arbitrary many vertices can cross in one crosscap.

3: Draw K_5 in the plane with one crosscap.



Orientable surface with k handles is usually denoted by S_k . Non-orientable surface with k cross-caps is usually denoted by N_k .

Let Σ be a surface obtained from the sphere by adding h handles and c cross-caps. Then the **Eulerian genus** of Σ is $2h + c$. Note that the textbook defines only orientable surfaces and defines genus of S_k to be k . We use Eulerian genus of S_k , which is $2k$. For surfaces, one handle is equivalent to 2 cross-caps (if there is another cross-cap - Torus is not the same as the Klein bottle)

Classification Theorem for Surfaces Any closed connected surface is homeomorphic to exactly one of the following surfaces: a sphere, a sphere with finitely many handles, or a sphere with a finitely many crosscaps glued in their place.

Orientable surfaces and Eulerian genus: S_0 plane (0), S_1 torus (2), S_2 double torus (4),...

Non-orientable surfaces and Eulerian genus: N_1 projective plane (1), N_2 Klein bottle (2), ...

Eulerian genus of a graph G is the smallest Eulerian genus of a surface where G can be embedded.

A region is called a **2-cell** if any closed curve can be continuously shrunk to a point (i.e no holes, handles, cross-caps). Embedding of a graph G in a surface is a **2-cell embedding** if every face is 2-cell.

Theorem - Euler for surfaces Let G be a graph embedded in a surface of Euler genus g . Then

$$|V(G)| + |F(G)| \geq |E(G)| + 2 - g,$$

where $F(G)$ is the set of faces of G . With equality if the embedding is a 2-cell embedding.

4: Let G be a graph embedded in a surface of genus g . Find an upper bound on the number of edges of G . (Hint: recall how we did it for planar graphs)

Notice that higher genus allows us to add slightly more edges.

Proof can be done by induction on the genus by cutting handles or cross-caps.

Similar as planar graphs, graphs embeddable to a surface of Eulerian genus g can be characterized by a finite set of forbidden minors. There are 35 forbidden minors for projective plane. Thousands for Torus... not all known.

- 5:** Draw K_4 in the projective plane such that every face is a 4-face (bounded by a 4-cycle).
- 6:** Draw Petersen's graph in the projective plane.
- 7:** Draw Petersen's graph on torus.
- 8:** Determine Eulerian genus of K_6
- 9:** Find embedding of K_7 in Torus.
- 10:** What is the largest n such that K_n can be embedded in the projective plane?
- 11:** Find two non-isomorphic embeddings of K_5 on Torus.
- 12:** Prove Euler for surfaces theorem.