Fall 2016, MATH-566

Interpretation of Duality and Duality theorem

Dualization for everyone:
$A \in \mathbb{R}^{m\times n}, \mathbf{c} \in \mathbb{R}^n, \mathbf{b} \in \mathbb{R}^m$

	primal	dual
variables	$x_1,\dots,x_n$	$y_1,\dots,y_m$
matrix	A	A^T
right hand	$\mathbf{b}$	$\mathbf{c}$
objective	max $\mathbf{c}^T\mathbf{x}$	min $\mathbf{b}^T\mathbf{y}$
constraint	ith constrain $\le$	$y_i \ge 0$
	ith constrain $\ge$	$y_i \le 0$
	ith constrain $=$	$y_i \in \mathbb{R}$
	$x_i \ge 0$	ith constrain $\ge$
	$x_i \le 0$	ith constrain $\le$
	$x_i \in \mathbb{R}$	ith constrain $=$

Diet problem: How much apricots (x_1), bananas (x_2) and cucumbers (x_3) one has to eat to get enough of Vit A, B, C? Minimize cost.

Need to know: % of daily value and cost:

	A	C	K	$	ammount
apricots	60	26	6	1.53	155g
bananas	3	33	1	0.37	225g
cucumbers	2	7	12	0.18	133g

1: Write Linear Program (P) solving the diet problem and write also its dual (D)

2: What are units of y_i in (D)? (Hint: inequalities need to make sense in units.)

3: Imagine you want to create a multivitamine pills ACK. What is the maximum price of one ACK pull if it has to deliver 100% of recommended daily value of vitamins A,C, and K and it must beat any fruit and vegetable in terms of price? *(If you don't manage to beat fruit and vegetable, nobody will buy your fancy ACK pill)* (Compute your price as a combination of prices of each of the vitamines.)

Duality Theorem

For the linear programs

$$\text{maximize } \mathbf{c}^T\mathbf{x} \text{ subject to } A\mathbf{x} \le \mathbf{b} \text{ and } \mathbf{x} \ge \mathbf{0} \qquad (P)$$

and

$$\text{minimize } \mathbf{b}^T\mathbf{y} \text{ subject to } A^T\mathbf{y} \ge \mathbf{c} \text{ and } \mathbf{y} \ge \mathbf{0} \qquad (D)$$

exactly one of the following possibilities occurs:

1. Neither (P) nor (D) has a feasible solution.
2. (P) is unbounded and (D) has no feasible solution.
3. (P) has no feasible solution and (D) is unbounded.
4. Both (P) and (D) have a feasiblesolution. Then both have an optimal solution, and if $\mathbf{x}^\star$ is an optimal solution of (P) and $\mathbf{y}^\star$ is an optimal solution of (D), then

$$\mathbf{c}^T\mathbf{x}^\star = \mathbf{b}^T\mathbf{y}^\star.$$

That is, the maximum of (P) equals the minimum of (D).

Next goal is to prove the duality theorem.