Fall 2016, MATH-566

Farkas Lemma and proof of duality

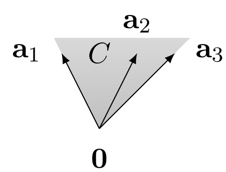
Farkas Lemma: Let $A \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Exactly one of the following holds

- $\exists \mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$
- $\exists \mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{y}^T A \geq \mathbf{0}^T$ and $\mathbf{y}^T\mathbf{b} < 0$

1: Is it possible to satisfy both conditions at the same time?

A (convex) **cone** is a set $C \in \mathbb{R}^d$ for which $\mathbf{x}, \mathbf{y} \in C$ and $a, b \geq 0$ implies $a\mathbf{x} + b\mathbf{y} \in C$.
A cone C generated by $X = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subseteq \mathbb{R}^d$ are are all linear combinations of vectors in X with nonnegative coefficients

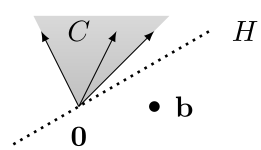
$$C = \{t_1\mathbf{a}_1 + t_2\mathbf{a}_2 + \cdots + t_n\mathbf{a}_n : t_i \geq 0\} \subseteq \mathbb{R}^d$$



Convex cone can be defined for any generating set X. If X is finite, then C is closed.

Geometric of Farkas Lemma: Let $\mathbf{a}_1, \ldots, \mathbf{a}_n, \mathbf{b} \in \mathbb{R}^m$. Let C be the convex cone generated by $\mathbf{a}_i$s. Exactly one of the following holds

- $\mathbf{b} \in C$
- exists hyperplane H such that $\mathbf{0} \in H$ and H strictly separates $\mathbf{b}$ from C. That is $H = \{\mathbf{x} : \mathbf{h}^T\mathbf{x} = 0\}$ and $\forall i, \mathbf{h}^T\mathbf{a}_i \geq 0$ and $\mathbf{h}^T\mathbf{b} < 0$.



2: Prove Farkas lemma using separation theorem. (What the separation gives?)

Reformulations of Farkas lemma:

- $A\mathbf{x} = \mathbf{b}$ has a non-negative solution iff $\forall \mathbf{y} \in \mathbb{R}^m$ with $\mathbf{y}^T A \geq \mathbf{0}^T$ also $\mathbf{y}^T\mathbf{b} \geq 0$.
- $A\mathbf{x} \leq \mathbf{b}$ has a non-negative solution iff $\forall \mathbf{y} \in \mathbb{R}^m$, $\mathbf{y} \geq \mathbf{0}$ with $\mathbf{y}^T A \geq \mathbf{0}^T$ also satisfies $\mathbf{y}^T\mathbf{b} \geq \mathbf{0}$.
- $A\mathbf{x} \leq \mathbf{b}$ has a solution iff $\forall \mathbf{y} \in \mathbb{R}^m$, $\mathbf{y} \geq \mathbf{0}$ with $\mathbf{y}^T A = \mathbf{0}^T$ also satisfies $\mathbf{y}^T\mathbf{b} \geq \mathbf{0}$.

Lets have linear programs

$$\text{maximize } \mathbf{c}^T\mathbf{x} \text{ subject to } A\mathbf{x} \leq \mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0} \tag{P}$$

$$\text{minimize } \mathbf{b}^T\mathbf{y} \text{ subject to } A^T\mathbf{y} \geq \mathbf{c} \text{ and } \mathbf{y} \geq \mathbf{0} \tag{D}$$

Lemma (Weak Duality): Let $\mathbf{x}$ and $\mathbf{y}$ be feasible solutions of (P) and (D). Then

$$\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y}$$

3: Prove the weak duality

Proof of the duality theorem point 4. from the Farkas lemma. ($\mathbf{c}^T\mathbf{x}^\star = \mathbf{b}^T\mathbf{y}^\star$.)
Let $\mathbf{x}^\star$ be optimal solution. Let $\gamma = \mathbf{c}^T\mathbf{x}$.

4: Are there solutions to $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{c}^T\mathbf{x} \geq \gamma$?

5: Are there solutions to $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{c}^T\mathbf{x} \geq \gamma + \varepsilon$, where $\varepsilon > 0$?

Let $\hat{A} = \binom{A}{-\mathbf{c}^T}$ and $\hat{\mathbf{b}}_\varepsilon = \binom{\mathbf{b}}{-\gamma-\varepsilon}$.

6: Apply Farkas Lemma on $\hat{A}\mathbf{x} \leq \hat{\mathbf{b}}_\varepsilon$ (which version?, write $\hat{\mathbf{y}}$ from FL as $(\mathbf{u}, z) \in \mathbb{R}^{m+1}$?)

7: What happens if $z = 0$? (Hint: Use Farkas lemma again with $\varepsilon = 0$.)