Fall 2016, MATH-566

Consequences of duality

Complementary slackness: Let

$$(P) \text{ min } \mathbf{c}^T\mathbf{x} \text{ s.t. } A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0$$

$$(D) \text{ max } \mathbf{b}^T\mathbf{y} \text{ s.t. } A^T\mathbf{y} \leq \mathbf{c}$$

Theorem Let $\mathbf{x}$ and $\mathbf{y}$ be feasible for (P) and (D). Vectors $\mathbf{x}$ and $\mathbf{y}$ are optimal solutions iff

- $x_i > 0 \Rightarrow \mathbf{y}^T\mathbf{a}_i = c_i$
- $x_i = 0 \Leftarrow \mathbf{y}^T\mathbf{a}_i < c_i$

where $\mathbf{a}_i$ is ith column of A.

1: Prove the theorem. Consider $(\mathbf{y}^T A - \mathbf{c}^T)\mathbf{x} =?$

Example: Diet problem. Suppose in optimal solution $\mathbf{a_i x} > b_i$. The $y_j = 0$. In optimal solution we get more than we need, so the cost in the dual is zero.

Geometric interpretation and solutions with complementary slackness

$$(P)\begin{cases} \text{min} & 18x_1 + 12x_2 + 2x_3 + 6x_4 \\ \text{s.t} & 3x_1 + x_2 - 2x_3 + x_4 = 2 \\ & x_1 + 3x_2 + 0x_3 - x_4 = 2 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{cases}$$

2: Write the dual and solve it using graphical method (draw half-spaces). The reconstruct the solution of (P).

Sensitivity Let

$$(P)\ \min\ \mathbf{c}^T\mathbf{x}\ \text{s.t.}\ A\mathbf{x}=\mathbf{b}, \mathbf{x}\geq 0$$

$$(D)\ \max\ \mathbf{b}^T\mathbf{y}\ \text{s.t.}\ A^T\mathbf{y}\leq \mathbf{c}$$

How does the solution change if $\mathbf{b}$ changes?

Which of the constraints are important and which are not?

Consider optimal solution $\mathbf{x}^\star = (\mathbf{x}_B, 0)$. Then $A = (B|\text{trash})$. Submatrix B is called the *base* of the solution. Note $\mathbf{x}_B = B^{-1}\mathbf{b}$.

$$\mathbf{c}^T\mathbf{x}^\star = \mathbf{c}_B^T\mathbf{x}_B = \mathbf{c}_B^T B^{-1}\mathbf{b}$$

Hence $(\mathbf{y}^\star)^T = \mathbf{c}_B^T B^{-1}$.

Suppose $\mathbf{b} \to (\mathbf{b}+\Delta\mathbf{b})$. If $\Delta\mathbf{b}$ small, base B is still the same. (see example) Then the new optimal solution is

$$B^{-1}(\mathbf{b}+\Delta\mathbf{b}) = \mathbf{x}_B + \Delta\mathbf{x}_B$$

3: What will be the change of the value of the objective function? (denoted by Δz)

$\mathbf{y}^\star$ gives sensitivity of the solution.

Let

$$(P) \min \mathbf{c}^T\mathbf{x} \text{ s.t. } A\mathbf{x} \geq \mathbf{b}, \mathbf{x} \geq 0$$
$$(D) \max \mathbf{b}^T\mathbf{y} \text{ s.t. } A^T\mathbf{y} \leq \mathbf{c}, \mathbf{y} \geq 0$$

Then complementary slackness gives

- $x_i > 0 \Rightarrow \mathbf{y}^T\mathbf{a}_i = c_i$
- $x_i = 0 \Leftarrow \mathbf{y}^T\mathbf{a}_i < c_i$
- $y_i > 0 \Rightarrow \mathbf{a}^i\mathbf{x} = b_i$
- $y_i = 0 \Leftarrow \mathbf{a}^i\mathbf{x} > b_i$,

where 'a^i is ith row of A and where $\mathbf{a}_i$ is ith column of A.