Fall 2016, MATH-566

Gomory-Hu Trees

Let $G = (V, E)$ be an **undirected** graph and $u : E \to \mathbb{R}_+$ be capacities on edges.

Problem: Compute minimum s-t-cut for all pairs $(s,t) \in V^2$.

Simple solution: Run $\binom{n}{2}$ times maximum-flow algorithm (it gives minimum cut too).

Better solution: Run $(n-1)$ times maximum-flow algorithm. Due to Gomory-Hu.

Denote the minimum capacity of s-t-cut by λ_{st}.

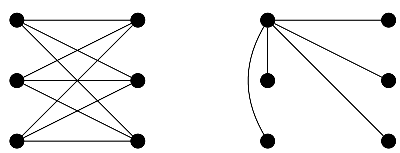
1: Show that for any $i, j, k \in V(G)$, $\lambda_{ik} \geq \min\{\lambda_{ij}, \lambda_{jk}\}$.

A tree T is a **Gomory-Hu Tree** for (G, u) if $V(T) = V(G)$ and $\forall s, t \in V$

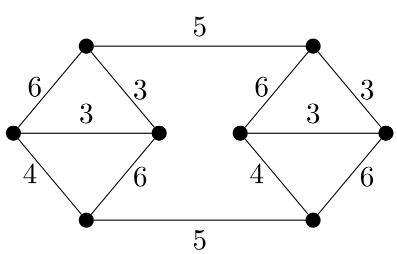
$$\lambda_{st} = \min_{e \in E(P_{st})} u(\delta_G(C_e)),$$

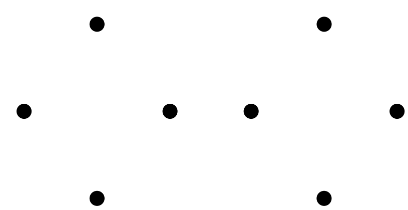
where P_{st} is the unique s-t-path in T, C_e is the set of vertices in the same connected component of $T - e$ as s and $\delta_G(C_e)$ is the cut defined by C_e in G.

Example of G, where $u : E \to 1$ is a constant. The tree T is then a star.



2: Find Gomory-Hu tree for the following graph with weights on edges.

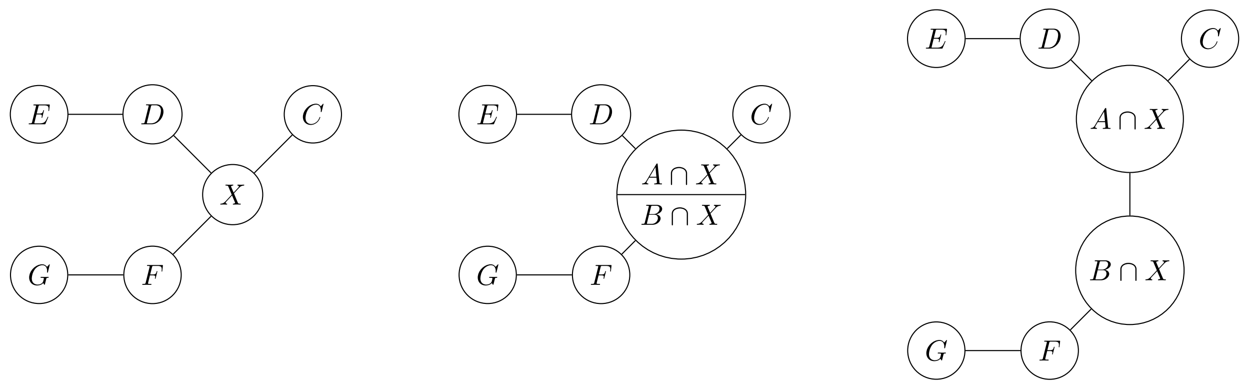




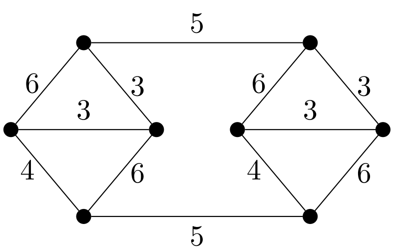
Algorithm:

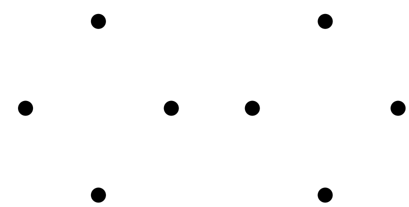
1. $T = (\{V\}, \emptyset)$
2. while exists $X \in V(T)$, where $|X| \geq 2$,
3. pick any s, t in X
4. contract vertices of all nodes other than X
5. find minimum s-t-cut $A \cup B = V$
6. replace X in $V(T)$ by edge $\{\{A \cap X\}, \{B \cap X\}\}$.

Sketch of one iteration:



3: Run the algorithm on the graph from question 2.

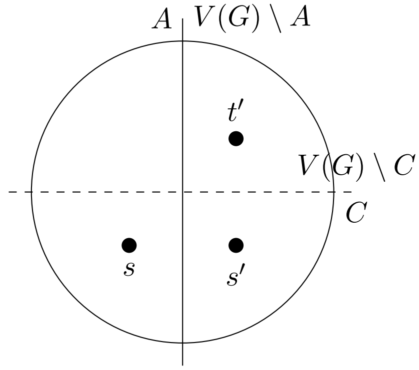




4: *(Cuts are submodular)* That is, let $A, B \subset V$. Show that

$$u(\delta(A \cup B)) + u(\delta(A \cap B)) \leq u(\delta(A)) + u(\delta(B)).$$

5: *Algorithm creates optimal cuts* Let $s, t \in V$ and let $A \subset V$ such that $\delta(A)$ is a minimum s-t-cut. Let $s', t' \in V \setminus A$. Let (G', u') be obtained from G by contracting vertices of A into one vertex a'. Let $K \subset (G')$ such that $\delta_{G'}(K \cup \{a'\})$ is a minimum s'-t'-cut in G'. Show that $\delta_G(K \cup A)$ is a minimum s'-t'-cut in G.
Proof beginning: Assume $\delta(C)$ is a minimum s'-t' cut in G. Show that $\delta(C \cup A)$ is also a minimum s'-t' cut in G. Wlog $s \in A \cap C$.

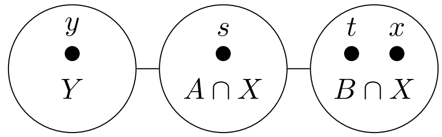


6: Let T be a tree during the run of algorithm. Let $e \in E(T)$ be any edge of T. Denote the endpoints of e by X and Y. Show that there are vertices $x \in X$ and $y \in Y$ such that e describes a minimum x-y cut.

The algorithm produces tree that works like Gomory-Hu tree at least for vertices adjacent in the tree.

Proof start: At the beginning of the algorithm (or after the first iteration, the observation is true. Let X and s, t be from step 3. The new edge $A \cap X$ to $B \cap X$ is correct due to s, t. Edges not incident with X are also correct. Remaining are edges that used to be incident with X but now are incident with $A \cap X$ (or $B \cap X$).

Suppose the edge YX had vertices y and x. If Y is in the A part of s-t-cut and x is in the other part, the edge $Y, (X \cap A)$ needs to be verified to satisfy the conclusion.



7: Show that the tree produced by the algorithm is indeed a Gomory-Hu tree.

8: *Gomory-Hu Tree* Construct Gomory-Hu Tree for the following graph using the algorithm from the class. Numbers on edges correspond to capacities.