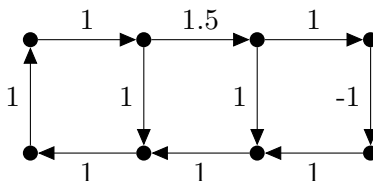


### Minimum Mean Cycle

Problem: Directed graph  $G = (V, E)$ , cost  $c : E \rightarrow \mathbb{R}$ . Find Minimum Mean Cycle  $C$ . That is,  $\frac{\sum_{e \in C} c(e)}{|C|}$  is minimized for  $C$ . Denote the minimum by  $\mu(G, c)$ .

1: Find minimum mean cycle in



*Walk* is a sequence of alternating vertices and edges  $v_1, e_1, v_2, e_2, \dots, v_k$  where edge  $e_i$  is edge  $\overrightarrow{v_i v_{i+1}}$ . *Length* of a walk is the number of edges in the walk.

Assume there is a vertex  $s$  such that every vertex of  $G$  is reachable from  $s$ .

Let  $F_k(x)$  be the minimum cost of a walk from  $s$  to  $x$  of length  $k$ . If no such walk exists,  $F_k(x) = \infty$ .

2: What happens if there is  $k$  such that  $F_k(x) = \infty$  for all  $x \in V$ ? If it happens for some  $k$ , what is the smallest  $k$  when it necessarily also happens?

**Solution:** There is no cycle. If there was a cycle, walk can cycle around a cycle and the walk can be infinitely long.

The smallest  $k$  is  $n$ . It may take  $n - 1$  steps to reach all vertices, consider a path.

Let  $C$  be the minimum mean cost cycle.

3: Let  $x \in C$  and  $F_k(x) < \infty$ . Compute upper bound on  $F_{k+|C|}(x)$ . Find sufficient conditions for  $\mu(G, c)$  and  $F_k(x)$  to make it tight.

**Solution:**

$$F_{k+|C|}(v) \leq F_k(v) + \sum_{e \in C} c(e)$$

Tight when  $F_k(x)$  is the least cost walk over all walks AND  $\mu(G, c) = 0$ .

Goal is to show

$$\mu(G, c) = \min_{x \in V} \max_{\substack{0 \leq k \leq n-1 \\ F_k(x) < \infty}} \frac{F_n(x) - F_k(x)}{n - k}$$

4: Assume  $\mu(G, c) = 0$ . Show that

$$0 = \min_{x \in V} \max_{\substack{0 \leq k \leq n-1 \\ F_k(x) < \infty}} \frac{F_n(x) - F_k(x)}{n - k}$$

by arguing that  $\leq$  is always true and that there exists a vertex that has equality.

**Solution:**  $\mu(G, c) = 0$  implies no negative circuit. Hence  $F_k(x)$  is  $\geq$  distance of  $x$  from  $s$ . Hence  $\max_k F_n(x) - F_k(x) \geq 0$ .

Let  $C$  be the minimum mean cost cycle. Let  $w \in C$ . Consider a shortest (least cost) path from  $s$  to  $w$  followed by  $n$  repetitions of  $C$ . This has the same cost as the path, so any initial part must be also least cost. Take first  $n$  steps of the path and this gives the desired  $x$ .

**5:** Assume  $\mu(G, c) = 0$ . Let  $\delta \in \mathbb{R}$  and let  $c' : E \rightarrow \mathbb{R}$  be defined as  $c'(e) = c(e) + \delta$ . ( $c'$  is adding  $\delta$  to cost of all edges) What is  $\mu(G, c')$  and if  $F'$  corresponds to  $c'$ , what is

$$\frac{F'_n(x) - F'_k(x)}{n - k}?$$

**Solution:** Let  $C$  be the minimum mean cycle. Then

$$\mu(G, c') = \mu(G, c) + \frac{\delta|C|}{|C|} = \mu(G, c) + \delta.$$

and

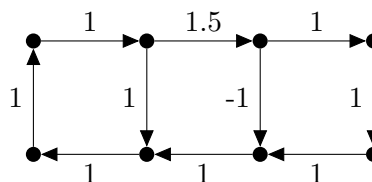
$$\frac{F'_n(x) - F'_k(x)}{n - k} = \frac{F_n(x) + n\delta - F'_k(x) - k\delta}{n - k} = \frac{F_n(x) - F'_k(x)}{n - k} + \delta$$

Since the change is the same for all cycles, we can add  $\delta$  and the solution would be the same. This leads to the following algorithm.

**Algorithm Minimum Mean Cycle:**

1. add vertex  $s$  and edges  $sv$  for all  $v \in V$  with  $c(sv) = 0$
2.  $F_0(s) := 0$ ;  $n := |V \cup \{s\}|$ ; and  $\forall v \in V, F_0(v) = \infty$ .
3. for  $k \in \{1, \dots, n\}$
4.       for all  $v \in V$
5.                $F_k(v) := \infty$
6.       for all  $\vec{uv} \in E$
7.               if  $F_k(v) > F_{k-1}(u) + c(uv)$  then
8.                        $F_k(v) := F_{k-1}(u) + c(uv)$  and  $p_k(v) := u$
9. if  $F_n(x) = \infty$  for all  $x \in V$ , then  $G$  is acyclic
10. Find  $x$  minimizing  $\max_{k, F_k(x) < \infty} \frac{F_n(x) - F_k(x)}{n - k}$ .
11. Minimum mean cycle is in  $\dots, p_{n-2}(p_{n-1}(p_n(x))), p_{n-1}(p_n(x)), p_n(x), x$

**6:** Run the algorithm on



**7:** What is the time complexity?

**Solution:**  $O(mn)$  We need  $n$  iterations and in each of them, each of  $m$  edges is used once.

*Next time: Integer Programming*