Fall 2016, MATH-566

Integer Programming - Unimodularity

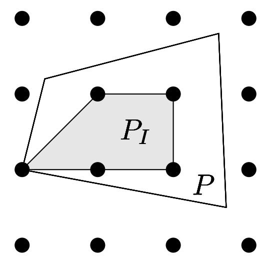
Source: Bill,Bill,Bill,Alex book, chapter 6.5

Problem:

$$(IP)\begin{cases}\text{maximize} & \mathbf{c}^T\mathbf{x}\\ \text{subject to} & A\mathbf{x}\le\mathbf{b},\end{cases}$$

where $\mathbf{c}\in\mathbb{Z}^n, \mathbf{b}\in\mathbb{Z}^m, A\in\mathbb{Z}^{m\times n}$, and $\mathbf{x}\in\mathbb{Z}^n$.

Let $P=\{\mathbf{x}\in\mathbb{R}^n : A\mathbf{x}\le\mathbf{b}\}$ be a polyhedron. Let $P_I = conv(\{\mathbf{x}\in\mathbb{Z}^n : A\mathbf{x}\le\mathbf{b}\})$ be the convex hull of integer points in P. If A and $\mathbf{b}$ are rational, P is called *rational polyhedra.*



Clearly, $P_I\subseteq P$. Polyhedron P is **integral** if $P=P_I$. (or if every face of P contains an integral vector) If P is integral, then (IP) can be solved as linear programming.

1: Why (IP) cannot be always solved by linear program and then rounding?

Theorem 6.22 Rational polytope P is integral iff for all $\mathbf{w}\in\mathbb{Z}^n$, the value of $\max\{\mathbf{w}^T\mathbf{x} : \mathbf{x}\in P\}$ is $\in\mathbb{Z}$.

2: Prove Theorem 6.22. One direction is easy. Other direction: Let $\mathbf{v}\in P$ be the unique optimal solution corresponding to $\mathbf{w}$.... show $\mathbf{v}$ has integer coordinates.

What guarantees and integral polyhedra?

Recall $A^{-1}=\frac{1}{\det(A)}A^\star$, where $A^\star_{i,j}=\det(A_{-i,-j})$.

For square matrices:

Theorem 6.23 Let $A\in\mathbb{Z}^{m\times m}$. Then $A^{-1}\mathbf{b}$ is integral for every $\mathbf{b}\in\mathbb{Z}^n$ iff $\det(A)\in\{1,-1\}$.

3: Prove Theorem 6.23

For rectangular matrix:

Matrix $A\in\mathbb{Z}^{m\times n}$ of full row rank is **unimodular** if every $m\times m$ basis of A (full rank square submatrix) has determinant ±1.

Theorem 6.24 Let $A \in \mathbb{Z}^{m\times n}$ be of full row rank. Polyhedron $P = \{\mathbf{x} : A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ is integral for every $\mathbf{b} \in \mathbb{Z}^m$ iff A is unimodular.

4: Prove Theorem 6.24

Matrix $A \in \mathbb{Z}^{m\times n}$ is **totally unimodular** if every square submatrix has determinant in $\{0, 1, -1\}$. (all entries of A are in $\{0, 1, -1\}$.)

HW question: A is totally unimodular iff $[A\ I]$ is unimodular (where I is $m \times m$ unit matrix).

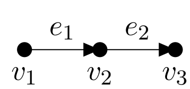
Theorem 6.25 Let $A \in \mathbb{Z}^{m\times n}$. Polyhedron $P = \{A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ is integral for every $\mathbf{b} \in \mathbb{Z}^m$ iff A is totally unimodular.

Theorem 6.26 Let $A \in \mathbb{Z}^{m\times n}$. Polyhedron $P = \{A\mathbf{x} \leq \mathbf{b}\}$ is integral for every $\mathbf{b} \in \mathbb{Z}^m$ iff A is totally unimodular.

Note: Matrix A being totally unimodular is decidable in polynomial time (algorithm by Seymour).

5: Let A have values $\{0, 1, -1\}$ and every column has at most one 1 and at most one -1. Show that A is totally unimodular. *Hint: induction.*

Example: Incidence matrix $M \in \mathbb{R}^{|V|\times|E|}$ of directed graph $G = (V, E)$ is totally unimodular. Matrix M is indexed by V and E. Edge $e = \overrightarrow{uv} \in E$ gives entries $M_{ue} = -1$ and $M_{ve} = 1$.



	e_1	e_2
v_1	-1	
v_2	1	-1
v_3		1

Theorem Matrix $A \in \mathbb{Z}^{m\times n}$ is totally unimodular iff for every $R \subseteq \{1, \dots, m\}$ there is a partition $R = R_1 \cup R_2$ such that for all $j \in 1 \dots n$.

$$\sum_{i\in R_1} a_{ij} - \sum_{i\in R_2} a_{ij} \in \{-1, 0, 1\}$$

Example: Incidence matrix $M \in \mathbb{R}^{|V|\times|E|}$ of (undirected) bipartite graph $G = (V, E)$ is totally unimodular. $M_{eu} = M_{ev} = 1$ for every $e = uv \in E$.

Next time: Integer Programming - Branch&Bound