

## Integer Programming - Solution *Methods* - Cutting Planes

Source: Bill, Bill, Bill;

[http://cgm.cs.mcgill.ca/~avis/courses/567/notes/cutplane\\_ex.pdf](http://cgm.cs.mcgill.ca/~avis/courses/567/notes/cutplane_ex.pdf)

<https://www.youtube.com/watch?v=YIeSwj4W6YI>

Problem:

$$(IP) \begin{cases} \text{maximize} & \mathbf{c}^T \mathbf{x} \\ \text{subject to} & \mathbf{Ax} \leq \mathbf{b}, \end{cases}$$

where  $\mathbf{c} \in \mathbb{Z}^n$ ,  $\mathbf{b} \in \mathbb{Z}^m$ ,  $A \in \mathbb{Z}^{m \times n}$ , and  $\mathbf{x} \in \mathbb{Z}^n$ . Notation:

$$P = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{Ax} \leq \mathbf{b}\} \qquad P_I = \text{conv}(\{\mathbf{x} \in \mathbb{Z}^n : \mathbf{Ax} \leq \mathbf{b}\})$$

Idea: Get NEW inequalities that better describe  $P_I$  (cut piece of  $P$  away). Main tool is  $\lfloor \cdot \rfloor$ .

Example:

$$4x_1 + 2x_5 \leq 5 \Rightarrow 2x_1 + x_5 \leq \frac{5}{2} \Rightarrow 2x_1 + x_5 \leq \left\lfloor \frac{5}{2} \right\rfloor = 2.$$

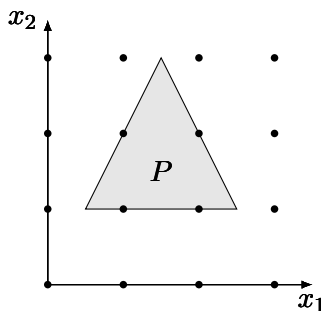
In general, for every  $\mathbf{u} \geq 0$ :

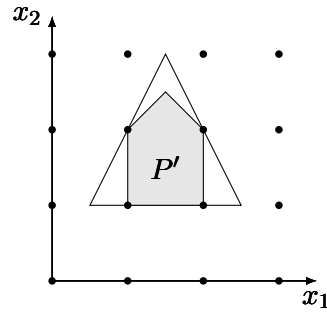
$$P_I \subseteq P' = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{u}^T \mathbf{Ax} \leq \lfloor \mathbf{u}^T \mathbf{b} \rfloor \text{ for all } \mathbf{u} \geq 0 \text{ with } \mathbf{u}^T A \text{ integral}\} \subseteq P$$

**Theorem:** It is sufficient to consider  $0 \leq \mathbf{u} \leq 1$ .

**1:** Find  $P'$  for the following set of inequalities:

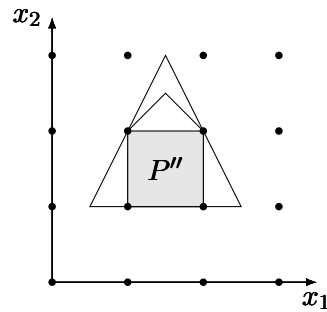
$$\begin{cases} 2x_1 + x_2 \leq 6 \\ -2x_1 + x_2 \leq 0 \\ -x_2 \leq -1 \end{cases}$$





2: Try to do the same operation on  $P'$  and obtain  $P''$ . Recall  $P'$  is given by:

$$\begin{array}{llll}
 2x_1 + x_2 \leq 6 & -2x_1 + x_2 \leq 0 & -x_2 \leq -1 & \\
 x_1 \leq 2 & -x_1 \leq -1 & -x_1 + x_2 \leq 1 & x_1 + x_2 \leq 4
 \end{array}$$



Notice  $P'' = P_I$ .

Make a sequence  $P = P^{(0)} \supseteq P' = P^{(1)} \supseteq P'' = P^{(2)} \supseteq \dots \supseteq P_I$ .

**Theorem** If  $P$  is a rational polytope, then there exists  $k$  such that  $P^{(k)} = P_I$ .

The smallest  $k$  such that  $P^{(k)} = P_I$  is called *Chvátal's rank*.

**How to generate cutting planes?** Run simplex algorithm and get cuts for things that are not integral.

Assume  $x_1, \dots, x_n \geq 0$  and integral. Constructing *Gomory Cut* for

$$a_1x_1 + \dots + a_nx_n = b \tag{1}$$

where  $a_j, b \in \mathbb{R}$  (not necessarily integral). Note that (1) can be written as

$$(\underbrace{[a_1] + (a_1 - [a_1])}_{f_1})x_1 + \dots + (\underbrace{[a_n] + (a_n - [a_n])}_{f_n})x_n = \underbrace{[b] + (b - [b])}_f,$$

$$([a_1] + f_1)x_1 + \dots + ([a_n] + f_n)x_n = [b] + f \tag{2}$$

$$[a_1]x_1 + \dots + [a_n]x_n \leq [b] + f \tag{3}$$

$$[a_1]x_1 + \dots + [a_n]x_n \leq [b] \tag{4}$$

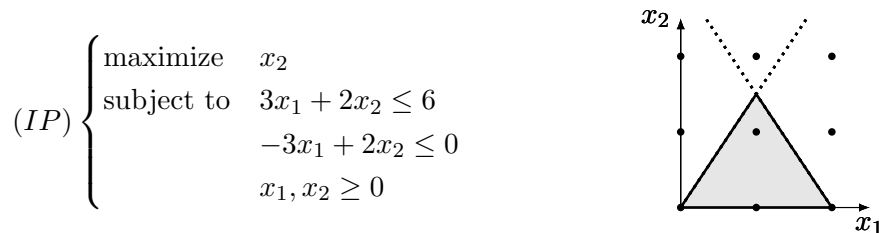
$$-[a_1]x_1 - \dots - [a_n]x_n \geq -[b] \tag{5}$$

$$f_1x_1 + \dots + f_nx_n \geq f. \tag{6}$$

Notice (3) is obtained from (2) by removing non-integral parts on the lefthand side. Since the lefthand side of (3) is an integer, we can make the righthand side an integer and obtain (4). By multiplying (4) by  $-1$  we obtain (5). Finally, (6) is obtained by adding (2) and (5).

This can be used in Simplex method if it gives a solution that is not integral.

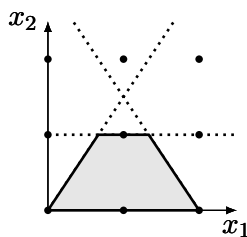
Example:



Solve LP relaxation using simplex method.

$$\begin{array}{rclcl} x_3 & = & 6 & - & 3x_1 & - & 2x_2 & & x_1 & = & 1 & - & \frac{1}{6}x_3 & + & \frac{1}{6}x_4 \\ x_4 & = & 0 & + & 3x_1 & - & 2x_2 & \sim & x_2 & = & \frac{3}{2} & - & \frac{1}{4}x_3 & - & \frac{1}{4}x_4 \\ z & = & 0 & & & + & x_2 & & z & = & \frac{3}{2} & - & \frac{1}{4}x_3 & - & \frac{1}{4}x_4 \end{array}$$

**3:** Find a cutting plane using  $x_2 = \frac{3}{2} - \frac{1}{4}x_3 - \frac{1}{4}x_4$ . Then substitute for  $x_3$  and  $x_4$  and get an inequality for the original problem.



Notice we got additional inequality. It is possible to add new slack variable  $x_5$  and add the following equation

$$\frac{1}{4}x_3 + \frac{1}{4}x_4 \geq \frac{1}{2} \Rightarrow \frac{1}{4}x_3 + \frac{1}{4}x_4 - x_5 = \frac{1}{2}$$

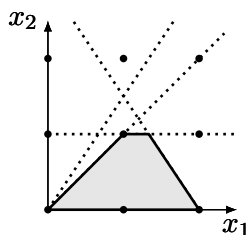
to the simplex table:

$$\begin{array}{rclcl} x_1 & = & 1 & - & \frac{1}{6}x_3 & + & \frac{1}{6}x_4 & & x_1 & = & 1 & - & \frac{1}{6}x_3 & + & \frac{1}{6}x_4 \\ x_2 & = & \frac{3}{2} & - & \frac{1}{4}x_3 & - & \frac{1}{4}x_4 & \Rightarrow & x_2 & = & \frac{3}{2} & - & \frac{1}{4}x_3 & - & \frac{1}{4}x_4 \\ z & = & \frac{3}{2} & - & \frac{1}{4}x_3 & - & \frac{1}{4}x_4 & & x_5 & = & -\frac{1}{2} & + & \frac{1}{4}x_3 & + & \frac{1}{4}x_4 \\ & & & & & & & & z & = & \frac{3}{2} & - & \frac{1}{4}x_3 & - & \frac{1}{4}x_4 \end{array}$$

Notice that the table is illegal since it assigns  $x_5 = -\frac{1}{2}$ . Notice we can reoptimize by changing  $x_3$  for  $x_5$ . We should actually use something called *Dual Simplex Method*. We get

$$\begin{aligned} x_1 &= \frac{2}{3} - \frac{2}{3}x_5 + \frac{1}{3}x_4 \\ x_2 &= 1 - x_5 \\ x_3 &= 2 + 4x_5 - x_4 \\ z &= 1 - x_5 \end{aligned}$$

4: Find another Gomory Cut.



Last simplex table is

$$\begin{array}{rcl} x_1 & = & \frac{2}{3} - \frac{2}{3}x_5 + \frac{1}{3}x_4 \\ x_2 & = & 1 - x_5 \\ x_3 & = & 2 + 4x_5 - x_4 \\ x_6 & = & -\frac{2}{3} + \frac{2}{3}x_5 + \frac{2}{3}x_4 \\ z & = & 1 - x_5 \end{array} \quad \sim \quad \begin{array}{rcl} x_1 & = & 1 - x_5 + \frac{1}{2}x_6 \\ x_2 & = & 1 - x_5 \\ x_3 & = & 1 + 5x_5 - \frac{3}{2}x_6 \\ x_4 & = & 1 - x_5 + \frac{2}{3}x_6 \\ z & = & 1 - x_5 \end{array}$$

Now the solution is integral.

- May need many cuts (but terminates if something like Bland's rule used)
- Used together with Branch and Bound