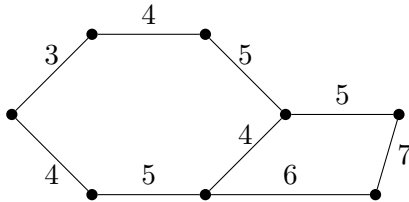


Minimum-Weight Perfect Matching in Bipartite Graphs

Source: Bill, Bill, Bill

Let $G = (V, E)$ be a graph. Let $c : E \rightarrow \mathbb{R}^+$ is a cost. Find a perfect matching M that is minimizing the sum of costs of the edges in the matching.

- 1: Find minimum-weight perfect matching in the following graph:



- 2: Write the minimum-weight perfect matching as an integer program (IP) on a graph $G = (V, E)$.

- 3: Consider a relaxation of (IP) to a linear program (P) and write the dual (D) of (P).

Theorem Birkhoff: If G is a bipartite graph, then solution to (P) is integral. (Why?)

- 4: Formulate complementary slackness conditions for optimal solution \mathbf{x} of (P) and optimal solution \mathbf{y} of (D).

Algorithm idea: Maintain an optimal solution to (D), create a solution to (P) whose value is matching the dual solution.

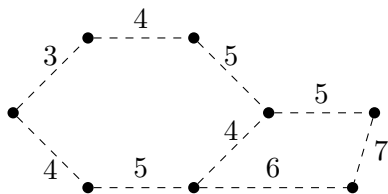
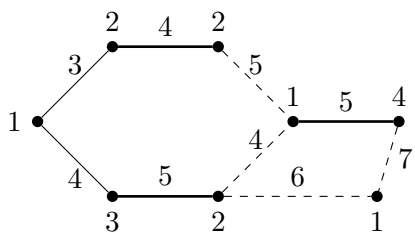
- 5: Find initial solutions to (P) and (D), where solution to (D) is feasible and the solutions satisfy complementary slackness. (Solution to (P) does not have to be feasible.)

6: If the solution to (D) is fixed, which edges can be used in matching? (Denote the edges by $E_=\$.)

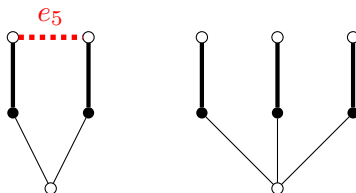
Algorithm sketch, suppose a perfect matching exists.

- start with initial solution \mathbf{x}, \mathbf{y} .
- Take edges $E_=\$ and try to find a perfect matching M by growing augmenting forests
- If M is not perfect, there are some outer of F vertices adjacent to edges in E but not in $E_=\$.
- Update \mathbf{y} to allow more edges in $E_=\$ and repeat.

7: What to do if there is no perfect matching in $E_=\$? Consider the following example. Number on edge e is $c(e)$, number at vertex v is y_v . How to modify \mathbf{y} to allow the tree to grow?



Recall that during growing the tree, we encountered edges like e_5 that were not possible to drop from the tree. The above algorithm does not work for edges with e_5 .



Our algorithm works only for **bipartite** graphs. Can be (nontrivially) generalized for all graphs. (The linear program has to be stronger by adding more constraints - for all odd vertex subsets, at least one edge is in the cut, blossoms need to be treated carefully.)