# Semidefinite Programming - Quick Intro

Source: Matoušek semidefinite programming

Recall: Let $A = \mathbb{R}^{n\times n}$. The *trace* of $A$ is $Tr(A) = \sum_{i=1}^{n} a_{i,i}$.

Let $SYM_n$ be symmetric matrices in $\mathbb{R}^{n\times n}$.

For $X, Y \in \mathbb{R}^{n\times n}$, let the dot product of $X$ and $Y$ be $X \bullet Y = Tr(X^T Y)$.

$X \in SYM_n$ is positive semidefinite if $v^T X v \geq 0$ for all $v \in \mathbb{R}^n$, denoted by $X \succeq 0$.

$$(LP)\begin{cases} \text{maximize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{cases} \quad \text{is equivalent to } (LP)\begin{cases} \text{maximize} & \mathbf{c}\bullet\mathbf{x} \\ \text{subject to} & \mathbf{a}_1\bullet\mathbf{x} = b_1 \\ & \mathbf{a}_2\bullet\mathbf{x} = b_2 \\ & \vdots \\ & \mathbf{a}_m\bullet\mathbf{x} = b_m \\ & \mathbf{x} \geq 0 \end{cases}$$

where $\mathbf{c}, \mathbf{x} \in \mathbb{R}^n$, $\mathbf{b}, \in \mathbb{R}^m$, $A \in \mathbb{R}^{m\cdot n}$, and $\mathbf{a}_i$ is the $i$th row of $A$.

One can view semidefinite programming ($SDP$) as a linear program with matrices instead of vectors.

$$(SDP)\begin{cases} \text{maximize} & C\bullet X \\ \text{subject to} & A_1\bullet X = b_1 \\ & A_2\bullet X = b_1 \\ & \vdots \\ & A_m\bullet X = b_m \\ & X \succeq 0 \end{cases}$$

Where $C, X, A_i \in SYM_n$ and $b_i \in \mathbb{R}$.

**1:** Compute

$$Tr\left(\begin{pmatrix} c_{11} & c_{12} \\ c_{12} & c_{22}\end{pmatrix}^T \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22}\end{pmatrix}\right) = \begin{pmatrix} c_{11} & c_{12} \\ c_{12} & c_{22}\end{pmatrix} \bullet \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22}\end{pmatrix} =$$

**2:** Show that the following is an equivalent form of $(SDP)$ up to some scaling.

$$(SDP)\begin{cases} \text{minimize} & \sum_{i\leq j} c_{i,j}x_{i,j} \\ \text{subject to} & \sum_{i\leq j} a_{i,j,k}x_{i,j} = b_k \quad \text{for } k = 1\ldots m \\ & X \succeq 0 \end{cases}$$

**3:** Write the following linear program as a semidefinite program

$$(LP)\begin{cases} \text{maximize} & 2x_1 + 3x_2 \\ \text{subject to} & x_1 + 2x_2 = 1 \\ & x_1 - x_2 \geq 2 \\ & x_1, x_2 \geq 0 \end{cases}$$

**4:** Write the following general linear program as a semidefinite program

$$(LP)\begin{cases} \text{maximize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{cases}$$

Dual form of $(SDP)$ is

$$(DSDP)\begin{cases} \text{minimize} & \mathbf{b}^T\mathbf{y} \\ \text{subject to} & y_1A_1 + y_2A_2 + \cdots + y_mA_n - C \succeq 0 \end{cases}$$

$(SDP)$ is *strictly feasible* if exists feasible $X$ which is positive definite ($X \succ 0$).
$(DSDP)$ is *strictly feasible* if exists feasible $\mathbf{y}$ such that $(\sum_i \mathbf{y}A_i) - C \succ 0$.

**Theorem: Strong duality of** $(SDP)$
If $(SDP)$ is strictly feasible and has an optimal solution of value $\gamma$, then $(DSDP)$ is feasible and has an optimal solution of value $\gamma$.

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**Theorem: Solvability of** $(SDP)$ **in polynomial time**
Let $(SDP)$ be feasible, set of feasible solutions $F$ bounded. Let $R \in \mathbb{N}$ be such that $R \geq \sqrt{Tr(X^TX)}$ for all $X \in F$ and $\varepsilon > 0$ be constants. Let $n$ be the size of binary encoding of $(SDP)$. Then in polynomial time in $n$ we can compute $X' \in F$ of value *optimum* $- \varepsilon$.

In other words, if no solution is not too big ($R$) and we are happy with $\varepsilon$ precision, we have a polynomial time algorithm.

Solution is using interior point methods. There exist free and efficient implementations CSDP and SDPA.

**5:** Write the following program $(P)$ as $(DSDP)$

$$(P) \begin{cases} \text{minimize} & \frac{(\mathbf{c}^T\mathbf{x})^2}{\mathbf{d}^T\mathbf{x}} \\ \text{subject to} & A\mathbf{x} + \mathbf{b} \geq 0 \end{cases}$$

where $\mathbf{d}^T\mathbf{x} \geq 0$ whenever $A\mathbf{x} + \mathbf{b} \geq 0$. (So the objective function is always $\geq 0$ and we do not have to worry about division by zero.)

**6:** Now we show that the requirement of $R$ for polynomial time solvability is indeed necessary. Consider the following constraint for $(DSDP)$. Show that $x_n$ is HUGE in any feasible solution.

$$\begin{pmatrix} 1 & 2 & & & & & & & \\ 2 & x_1 & & & & & & & \\ & & 1 & x_1 & & & & & \\ & & x_1 & x_2 & & & & & \\ & & & & 1 & x_2 & & & \\ & & & & x_2 & x_3 & & & \\ & & & & & & \ddots & & \\ & & & & & & & 1 & x_{n-1} \\ & & & & & & & x_{n-1} & x_n \end{pmatrix} \succeq 0$$

Use that the matrix is positive semidefinite if each block is positive semidefinite.