Fall 2016, MATH-566

Graph Theory - Quick Run Trough Definitions

A **simple graph** G is an ordered pair (V, E) of **vertices** V and **edges** E, where $E \subseteq \{\{u, v\} : u, v \in V, u \neq v\}$.

$|V|$ is **order** of G

$|E|$ is **size** of G

Vertices of G are denoted by $V(G)$ and edges of G by $E(G)$.

If $\{u, v\} \in E$, then u and v are **adjacent** and called **neighbors**.

If $u \in V$ and $e \in E$ satisfy $v \in e$, then v and e are **incident**.

$\{u, v\}$ can be simplified to uv.

Edges are **adjacent** if they share vertices.

Drawing of G *assigns point* to V and *curves* to E, where endpoints of uv are u and v.

If $V(G) = \emptyset$ then G is a *null* graph.

Graph H is **subgraph** of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, notation $H \subseteq G$.

H is a **proper subgraph** if $H \subseteq G$, $H \neq G$ (and H is not a null graph).

H is a **spanning subgraph** if $H \subseteq G$ and $V(H) = V(G)$

H is a **induced subgraph** if $H \subseteq G$ and $\forall u, v \in V(H), uv \in E(G) \Rightarrow uv \in E(H)$.

If $X \in V(G)$, then $G[X]$ denotes induced subgraph H of G where $V(H) = X$.

We use $+$ and $-$ to denote adding edges or vertices to graph.

Walk in a graph G is a sequence $v_1, e_1, v_2, e_2, v_3, \ldots, v_n$, where $v_i \in V(G)$ and $e_i \in E(G)$, where consecutive entries are incident.

Trail is a walk without repeated edges.

Path is a walk without repeated vertices.

If walk,trail,path starts with u and ends with v, it is called $u - v$ walk,trail,path.

Length of a walk,trail,path is the number of edges.

Theorem 1.6 If a graph G contains $u - v$ walk, it also contains $u - v$ path.

Distance of u and v is the length of a shortest $u - v$ path, denoted $d(u, v)$.

Diameter of G, denoted by $diam(G)$ is maximum of $d(u, v)$ over all $u, v \in V$.

Walk/Trail is *closed* if it is $u - u$ walk/trail. Otherwise it is *open*.

Circuit is a closed trail.

Closed trail with no repetition of vertices (except first and last) is **cycle**.

Graph is **conneted** if for all $u, v \in V$ exists $u - v$ walk.

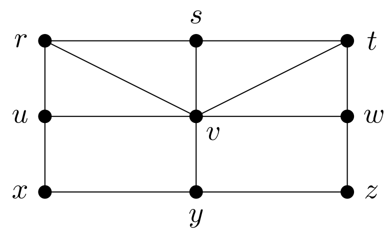
If graph in not connected, it is **disconneted**.

Connected **component** of G is a connected subgraph of G that is not a proper subgraph of any other connected subgraph of G.

Graph G is a **union** of graph $G_1, \ldots, G_k$ if G can be partitioned into $G_1, \ldots, G_k$. Notation $G = G_1 \cup G_2 \cup \cdots \cup G_k$.

1: *1.3* Let $S = \{2, 3, 4, 7, 11, 13\}$. Draw the graph G whose vertex set is S and such that $ij \in E(G)$ for all $i, j \in S$ if $i + j \in S$ or $|i - j| \in S$. What is $|E(G)|$ and $|V(G)|$? What is diameter of G?

2: For the depicted graph G, give an example of each of the following or explain why no such example exists.

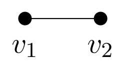


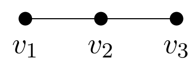
(a) An $x - y$ walk of length 6.
(b) A $v - w$ trail that is not a $v - w$ path.
(c) An $r - z$ path of length 2.
(d) An $x - z$ path of length 3.
(e) An $x - t$ path of length $d(x,t)$.
(f) A circuit of length 10.
(g) A cycle of length 8.
(h) A geodesic whose length is $diam(G)$.

3: *1.15* Draw all connected graphs of order 5 in which the distance between every two distinct vertices is odd. Explain why you know that you have drawn all such graphs.

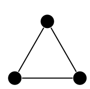
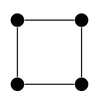
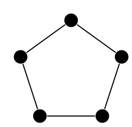
Path P_n of length $n-1$ has vertices $v_1, \ldots, v_n$ and edges $v_i v_{i+1}$ for all $1 \le i \le n-1$.

P_1: 

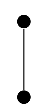
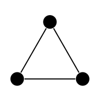
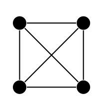
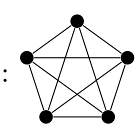
P_2: 

P_3: 

Cycle C_n of length n if obtained from $P_n = v_1, \ldots, v_n$ by adding edge $v_1 v_n$

C_3: 
C_4: 
C_5: 

Complete graph K_n has n vertices and for all $u, v \in V(K_n), uv \in E(K_n)$, i.e. *all* edges.

K_1: 
K_2: 
K_3: 
K_4: 
K_5: 

4: What is $|E(K_n)|$?

The **complement** $\overline{G}$ of a graph G is graph where $V(\overline{G}) = V(G)$ and $uv \in E(G)$ iff $uv \notin E(\overline{G})$.

Complement of complete graph is **empty** graph (or **independent set**).

Theorem 1.11 If G is disconnected then $\overline{G}$ is connected.

Graph G is **bipartite** if $V(G) = X \cup Y$, where $G[X]$ and $G[Y]$ are empty graphs.

Theorem 1.12 Graph G is bipartite iff G does not contain an odd cycle.

Complete bipartite graph $K_{m,n}$ is a bipartite graph with parts $|V_1| = m$ and $|V_2| = n$ and for all $u \in V_1$ and $v \in V_2$ we have $uv \in E(K_{m,n})$.

$K_{1,n}$ is called a **star**.

Multigraph is a graph where edges can have multiplicities (**multiedges**) and **loops** (edge vv).

Directed graph (or digraph) has edges as ordered pairs rather then sets of size two.

Oriented graph is a graph where edges are oriented (directed).

5: What is the difference between directed graph and oriented graph?

Hypergraph is a graph where edges are any subsets of vertices (not just size 2).

Degree of a vertex v is the number of edges incident with v (loop counts $2\times$), denoted by $deg(v)$ or $d(v)$.

In digraph we count **in-degree** $d^-(v)$ and **out-degree** $d^+(v)$.

Neighborhood of a vertex v is the set of vertices adjacent to v, denoted by $N(v)$.

Note $deg(v) = |N(v)|$ for *simple* graphs.

Vertex v is **isolated** if $d(v) = 0$.

Vertex v is **leaf** if $d(v) = 1$.

The **minimum degree** of G is $\delta(G) = \min_{v \in V(G)} d(v)$.

The **maximum degree** of G is $\Delta(G) = \max_{v \in V(G)} d(v)$.

Theorem 2.1 If a graph G has m edges when

$$\sum_{v \in V(G)} deg(v) = 2m$$

A vertex of even degree is called an **even vertex**, while a vertex of odd degree is an **odd vertex**.

Corollary 2.3 Every graph has an even number of odd vertices.