Shortest path and linear programming

Shortest path
Input: Graph $G = (V, E)$, costs $c : E \to \mathbb{R}$, and $s, t \in V$.
Output: s-t-path P, where $\sum_{e \in P} c(e)$ is minimized.

Suppose every edge has an orientation (direction). This gives a directed graph. Any ordinary graph can be converted to a directed graph by adding two edges in opposite directions.

1: Create a linear program solving the shortest path problem. Hints: Minimize, overall cost, for every edge decide if it is in the path or not, make sure that the path starts at s (and ends at t). Make sure that the path does not stop at any other vertex (use that edges are oriented and you know incoming and leaving edges).

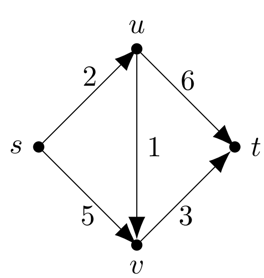
Solution:

$$\begin{cases} \text{minimize} & \sum_{e \in E} c(e) \cdot x_e \\ \text{subject to} & \sum_{(s,v) \in E} x_{s,v} = 1 \\ & -\sum_{(v,t) \in E} x_{v,t} = -1 \\ & \sum_{(v,w) \in E} x_{v,w} - \sum_{(u,v) \in E} x_{u,v} = 0 \text{ for all } v \neq s, t \\ & x_e \geq 0 \text{ for all } e \in E \end{cases}$$

Note: This uses the assumption that there is no edge going to s and no edge going from t. If they are there, it may happen that instead of path we get two cycles: one around s and one around t.
More correct formulation (without exceptions):

$$\begin{cases} \text{minimize} & \sum_{e \in E} c(e) \cdot x_e \\ \text{subject to} & \sum_{(s,v) \in E} x_{s,v} - \sum_{(v,s) \in E} x_{v,s} = 1 \\ & -\sum_{(t,v) \in E} x_{t,v} - \sum_{(v,t) \in E} x_{v,t} = -1 \\ & \sum_{(v,w) \in E} x_{v,w} - \sum_{(u,v) \in E} x_{u,v} = 0 \text{ for all } v \neq s, t \\ & x_e \geq 0 \text{ for all } e \in E \end{cases}$$

2: Write the linear program for graph with directed edges $E = \{su, sv, uv, ut, vt\}$, where the costs are $c(su) = 2, c(sv) = 5, c(uv) = 1, c(ut) = 6, c(vt) = 3$.



Solution:

$$\begin{cases} \text{minimize} & 2x_{su} + 5x_{sv} + x_{uv} + 6x_{ut} + 3x_{vt} \\ \text{subject to} & x_{su} + x_{sv} = 1 \\ & -x_{ut} - x_{vt} = -1 \\ & -x_{su} + x_{uv} + x_{ut} = 0 \\ & -x_{sv} - x_{uv} + x_{vt} = 0 \\ & x_e \geq 0 \text{ for all } e \in E \end{cases}$$

3: Write the dual linear program for shortest path. If confused, try it first for the particular graph and see how to generalize it.

Solution:

$$\begin{cases} \text{maximize} & y_s - y_t \\ \text{subject to} & y_u - y_v \leq c(uv) \text{ for all } (u,v) \in E \end{cases}$$

We can also add $y_s = 0$ to make it easier. The solutions are called feasible potential. We are trying to make y_t as negative as possible.

4: Interpret the dual program.

Solution: Number for every vertex, where we want to push y_t away from s as much as we can.

5: Does the LP work for negative costs? Why?

Solution: No.