Integer Programming - Unimodularity

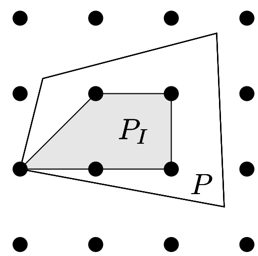
Source: Bill, Bill, Bill, Alex book, Chapter 6.5

Problem:

$$(IP)\begin{cases}\text{maximize} & \mathbf{c}^T\mathbf{x}\\ \text{subject to} & A\mathbf{x}\le\mathbf{b},\end{cases}$$

where $\mathbf{c}\in\mathbb{Z}^n, \mathbf{b}\in\mathbb{Z}^m, A\in\mathbb{Z}^{m\times n}$, and $\mathbf{x}\in\mathbb{Z}^n$.

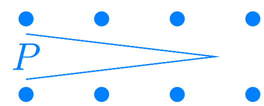
Let $P=\{\mathbf{x}\in\mathbb{R}^n : A\mathbf{x}\le\mathbf{b}\}$ be a polyhedron. Let $P_I=conv(\{\mathbf{x}\in\mathbb{Z}^n : A\mathbf{x}\le\mathbf{b}\})$ be the convex hull of integer points in P. If A and $\mathbf{b}$ are rational, P is called a **rational polyhedra**.



Clearly, $P_I\subseteq P$. The polyhedron P is **integral** if $P=P_I$. (or if every face of P contains an integral vector) If P is integral, then (IP) can be solved as linear programming.

1: What can go wrong if we try to solve (IP) by solving the associated linear program and rounding its solution?

Solution: Rounding may not be possible.



Theorem 6.22 A rational polytope P is integral iff for all $\mathbf{w}\in\mathbb{Z}^n$, the value of $\max\{\mathbf{w}^T\mathbf{x} : \mathbf{x}\in P\}$ is $\in\mathbb{Z}$.

2: Prove Theorem 6.22. One direction is easy. Other direction: Let $\mathbf{v}\in P$ be the unique optimal solution corresponding to $\mathbf{w}$ and show $\mathbf{v}$ has integer coordinates.

Solution: By multiplying $\mathbf{w}$ by a constant, assume that for all other vertices $\mathbf{u}\neq\mathbf{v}$:

$$\mathbf{w}^T\mathbf{v}>\mathbf{w}^T\mathbf{u}+\mathbf{u}_1-\mathbf{v}_1.$$

Hence $\mathbf{v}$ is optimal also for $\mathbf{z}=(\mathbf{w}_1+1,\mathbf{w}_2,\ldots)$. Then $\mathbf{z}^T\mathbf{v}=\mathbf{w}^T\mathbf{v}+\mathbf{v}_1$ Since we assumed $\mathbf{z}^T\mathbf{v}$ and $\mathbf{w}^T\mathbf{v}$ are integral, also $\mathbf{v}_1$ is integral. Repeat for other components of $\mathbf{v}$.

What guarantees and integral polyhedra?

Recall $A^{-1}=\frac{1}{\det(A)}A^\star$, where $A^\star_{i,j}=\det(A_{-i,-j})$.

For square matrices:

Theorem 6.23 Let $A\in\mathbb{Z}^{m\times m}$. Then $A^{-1}\mathbf{b}$ is integral for every $\mathbf{b}\in\mathbb{Z}^n$ iff $\det(A)\in\{1,-1\}$.

3: Prove Theorem 6.23

Solution: $\Leftarrow$ Let $\det(A)=\pm1$. By Cramer's rule, also A^{-1} is integral. Hence $A^{-1}\mathbf{b}$ is integral.
$\Rightarrow$ If $\mathbf{b}$ is ith unit vector, then $A^{-1}\mathbf{b}$ is ith column of A^{-1}. Hence A^{-1} is integral and

$\det(A^{-1})$ is an integer. Since $1 = \det(AA^{-1}) = \det(A) \cdot \det(A^{-1})$ and $\det(A) \in \mathbb{Z}$ we conclude that $\det(A) = \det(A^{-1}) = \pm 1$.

For rectangular matrices:
We say any matrix $A \in \mathbb{Z}^{m \times n}$ of full row rank is **unimodular** if every $m \times m$ basis of A (full rank square submatrix) has determinant ± 1.

Theorem 6.24 Let $A \in \mathbb{Z}^{m \times n}$ be of full row rank. Tbe polyhedron $P = \{\mathbf{x} : A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ is integral for every $\mathbf{b} \in \mathbb{Z}^m$ if and only if A is unimodular.

4: Prove Theorem 6.24

Solution: Recall in LP: A solution $\mathbf{x}$ is called a basic feasible solution if $\mathbf{x}$ has at most m non-zero entires and the columns of A corresponding to these entries are linearly independent.

$\Leftarrow$ Let $\overline{\mathbf{x}} \in P$ be a vertex (using $\mathbf{x} \geq 0$). Pick basis B corresponding to $\overline{\mathbf{x}}$ by picking columns where $\overline{\mathbf{x}}$ is nonzero and extend. Use Theorem 6.23 on $B\overline{\mathbf{x}} = \mathbf{b}$.

$\Rightarrow$ Let B be a base of A and pick any $\mathbf{v} \in \mathbb{Z}^n$. Our goal is to show that $B^{-1}\mathbf{v} \in \mathbb{Z}^m$ since then Theorem 6.23 implies $\det(B) = \pm 1$. Choose $\mathbf{y} \in \mathbb{Z}^m$ such that $B^{-1}\mathbf{v} + \mathbf{y} \geq \mathbf{0}$. Let $\mathbf{b} = B(B^{-1}\mathbf{v} + \mathbf{y}) = \mathbf{v} + B\mathbf{y} \in \mathbb{Z}^m$. Add zero components to $(B^{-1}\mathbf{v} + \mathbf{y})$, which gives $\mathbf{z} \in \mathbf{R}^n$ such that $A\mathbf{z} = \mathbf{b}$. Now $\mathbf{z} \in P$ and it corresponds to a basic feasible solution, hence $\mathbf{z} \in \mathbb{Z}^n$. Therefore $B^{-1}\mathbf{v} \in \mathbb{Z}^m$.

We say any matrix $A \in \mathbb{Z}^{m \times n}$ is **totally unimodular** if every square submatrix has determinant in $\{0, 1, -1\}$. In particular, all entries of A are in $\{0, 1, -1\}$.

HW question: A is totally unimodular iff $[A\ I]$ is unimodular (where I is $m \times m$ unit matrix).

Theorem 6.25 Let $A \in \mathbb{Z}^{m \times n}$. The polyhedron $P = \{A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ is integral for every $\mathbf{b} \in \mathbb{Z}^m$ iff A is totally unimodular.

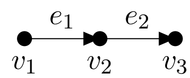
Theorem 6.26 Let $A \in \mathbb{Z}^{m \times n}$. The polyhedron $P = \{A\mathbf{x} \leq \mathbf{b}\}$ is integral for every $\mathbf{b} \in \mathbb{Z}^m$ iff A is totally unimodular.

Note: An algorithm of Seymour decides whether a matrix A is totally unimodular in polynomial time.

5: Let A have values $\{0, 1, -1\}$ and every column has at most one 1 and at most one -1. Show that A is totally unimodular. *Hint: induction.*

Solution: Let N be a $k \times k$ submatrix. If $k = 1$ clear. If column with at most one non-zero, expand the determinant. If all columns have 1 and -1, matrix is signular.

6: Show that the incidence matrix $M \in \mathbb{R}^{|V|\times|E|}$ of directed graph $G = (V, E)$ is totally unimodular. Matrix M is indexed by V and E. Edge $e = \overrightarrow{uv} \in E$ gives entries $M_{ue} = -1$ and $M_{ve} = 1$.



	e_1	e_2
v_1	-1	
v_2	1	-1
v_3		1

Solution: Note that each column of M has exactly one 1 and exactly one -1 entry. Moreover, all other entries are 0. By **5**, A is totally unimodular.

Theorem A matrix $A \in \mathbb{Z}^{m\times n}$ is totally unimodular iff for every $R \subseteq \{1, \dots, m\}$ there is a partition $R = R_1 \cup R_2$ such that for all j, $1 \le j \le n$.

$$\sum_{i\in R_1} a_{ij} - \sum_{i\in R_2} a_{ij} \in \{-1, 0, 1\}$$

7: Show that the incidence matrix $M \in \mathbb{R}^{|V|\times|E|}$ of an (undirected) bipartite graph $G = (V, E)$ is totally unimodular. $M_{ue} = M_{ve} = 1$ for every $e = uv \in E$.

Solution: For any $R \subseteq V(G)$, partition R according to any bipartition $A \cup B$ of $V(G)$, i.e., $R_1 := R \cap A$ and $R_2 := R \cap B$. Then for any $e \in E(G)$, write $e = ab$ where $a \in A$ and $b \in B$. Then,

$$\sum_{u\in R_1} m_{ue} - \sum_{u\in R_2} m_{ue} = \begin{cases} 0, & a, b \notin R \text{ or } a, b \in R \\ 1, & a, b \notin R \text{ or } a \in R \text{ and } b \notin R \\ -1, & a, b \notin R \text{ or } a \notin R \text{ and } b \in R \end{cases}$$

so by the above theorem, we are done.