This HW is due Mar 22 BEFORE the class starts. Submit electronically to the instructor. Choose your number, priority of numbers first come first served.

Homework 2
Participant number XXXX

1: Find the extremal number (with error term allowed) of the graph the icosahedron and of the Peterson graph.

Hint: Just need the chromatic number.

Solution:

2: Let $\mathrm{ex}(n, F_1, F_2, \ldots, F_t)$ be the maximum number of edges in an n-vertex graph containing no copy of any graph F_i. Determine the asymptotically sharp upper bound for $\mathrm{ex}(n, F_1, F_2, \ldots, F_t)$.

Solution:

3: For a fixed graph F, show that the following function is decreasing as n increases

$$\frac{\mathrm{ex}(n, F)}{\binom{n}{2}}.$$

Solution:

Theorem 1 (Füredi, 2015)**.** *Suppose G is an n-vertex K_{k+1}-free graph with $e(G) = e(T_k(n)) - t$. Then G can be obtained from a complete k-partite graph (on n vertices) by adding and removing at most $3t$ total edges.*

4: Prove Theorem 1.

Hint: We had some Lemma that might be useful.

Solution:

Theorem 2 (Erdős; Kővari-Sós-Turán, 1954)**.** *For any naturals $s \leq t$ we have*

$$\mathrm{ex}(n, K_{s,t}) \leq \frac{1}{2}(t-1)^{1/s} n^{2-1/s} + O(n).$$

5: Prove KST Theorem.

Hint: Let G be an n-vertex graph with no $K_{s,t}$ subgraph. Count the pairs (v, S) where v is a vertex and S is a set of s vertices in the neighborhood of v. Use this to finish the proof.

Solution:

6: Let S be a set of n points in the plane. Show that there are at most $O(n^{3/2})$ pairs of points that are of unit distance from each other[1].

Hint: Consider a graph on the given points and connect two vertices by an edge if they are at distance 1.

Solution:

7: Let $A \subset [n]$ be a set of integers such that no element divides the product of any other two elements. Prove that $|A| \leq \pi(n) + n^{2/3}$ where $\pi(n)$ is the number of primes not exceeding n.

Hint: Look at Bollobás *Modern Graph Theory*, exercises in the extremal chapter.

Solution:

[1] The best known upper bound is $O(n^{4/3})$ and is conjectured to be $n^{1+o(1)}$.