Homework 3

1: Show that there exists a graph F of chromatic number 3 and constant C (depending on F) such that

$$\mathrm{ex}(n, F) \geq \frac{1}{4}n^2 + Cn^{1.99}.$$

Hint: Prove instead $\mathrm{ex}(n, F) \geq \frac{1}{4}n^2 + Cn^{2-2/t}$ for any t. Let F be the complete tripartite graph with classes of size t.

2: Let Q_d be the graph of the d-dimensional hypercube. Show that the Ramsey number of Q_d is

$$R(Q_d, Q_d) \leq 2^{3d}.$$

Hint: Use dependent random choice on the graph spanned by the more common color.

3: Find another extremal graph for the path on k vertices (that is not $n/(k-1)$ copies of K_{k-1}) and confirm that it has the maximal number of edges.

Hint: The construction is connected and has a large independent set.

4: [Half Graph] Let H be a bipartite graph with classes $A = [n]$ and $B = [n]$. A pair of vertices $a \in A$ and $b \in B$ form an edge if and only if $a \geq b$.

(a) Find an explicit ϵ-regular partition of H into r parts where $3 \leq r \leq 10/\epsilon$.

(b) Show that for $\epsilon > 0$ small enough there exists $c > 0$ such that any ϵ-regular equipartition of H into r classes has at least cr many pairs of classes that are not ϵ-regular.

5: Use the regularity lemma to prove the following alternate version:

Given $\epsilon > 0$ and $m \geq 1$, there exists a constant $M = M(\epsilon, m)$ such that for every graph on n vertices (for n large enough) has a partition into $r+1$ parts $V_0, V_1, V_2 \ldots V_r$ such that $|V_0| \leq \epsilon n$ and $|V_1| = |V_2| = \cdots = |V_r|$ and all but at most ϵr^2 pairs of classes from $V_1, V_2, \ldots, V_r$ are ϵ-regular and $m \leq r < M$.

Hint: Use regularity lemma from class to get an $\epsilon/2$-regular partition and trim the too-big partition classes.

6:

Theorem 1 (Solymosi, 2001)**.** *For any $\alpha > 0$, there exists N such that if A is a set of $\geq \alpha N^2$ many points on the $N \times N$ integer lattice, then A contains three distinct points of the following form $(x, y), (x+d, y), (x, y+d)$, i.e., an isosceles right triangle.*

Proof. Consider the collection of vertical lines, horizontal lines and 45° diagonal (left to right) lines in the $N \times N$ lattice.

Construct a 3-partite graph G with classes X, Y, Z such that X is the set of vertical lines, Y is the set of horizontal lines, and Z is the set of diagonal lines.

Two vertices (lines) in this graph are connected by an edge if the intersection of the two lines is an element of A. Therefore, for each element $a \in A$, the three lines intersecting in a form a triangle in G.

Finish the proof by looking at triangles. □

7: Let G be a graph on n vertices that consists of the union of n induced matchings. Show that $e(G) = o(n^2)$.

Hint: Suppose for contradiction that G has at least δn^2 edges. Use regularity lemma. In the clan up, for each matching M also remove edges of M incident to V_i if there are less than $\epsilon|V_i|$ many of these edges. What happens if some edges remain? Final contradiction can be obtained that some matching is not induced.