Mantel's Theorem

Theorem 1 (Mantel's Theorem, 1907)**.** *The maximum number of edges in a graph on n vertices with no triangle subgraph is* $\lfloor \frac{n^2}{4} \rfloor$.

1: Show that the n-vertex complete balanced bipartite graph has $\lfloor \frac{n^2}{4} \rfloor$ edges. It means that the bound in Mantel's theorem is achieved by some graphs.

Solution: Observe that the n-vertex complete bipartite graph with class sizes $\lceil \frac{n}{2} \rceil$ and $\lfloor \frac{n}{2} \rfloor$ has no triangle subgraph and has exactly $\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor = \lfloor \frac{n^2}{4} \rfloor$ edges.

Now we show that there are no triangle-free graphs with more edges than claimed by Mantel's theorem.

2: Prove Mantel's theorem by induction, where the induction step removes two adjacent vertices.

Solution: Induction on n. If $n = 1, 2$ we are done, so assume $n > 2$ and that the statement of the theorem holds for smaller graphs. Let G be a triangle-free graph on n vertices and let xy be an edge of G. The graph $G - xy$ is obviously triangle-free and has $n - 2$ vertices, so it has at most $\lfloor \frac{(n-2)^2}{4} \rfloor$ edges by induction. The edge xy has at most $n - 2$ edges incident (otherwise there is a triangle). Thus G has at most $1 + (n - 2) + \frac{(n-2)^2}{4} = \frac{n^2}{4}$ edges.

3: If G is a triangle-free graph, then adjacent vertices have no common neighbors. So for an edge xy we have $d(x) + d(y) \leq n$ (don't forget to count the edge xy twice!). Use it in the following equation (and argue the equation is right)

$$\sum_{x \in V(G)} d(x)^2 = \sum_{xy \in E(G)} (d(x) + d(y)), \tag{1}$$

where d denotes the degree of a vertex. Then combine (1) with Cauchy-Schwartz

$$\left(\sum_i a_i b_i\right)^2 \leq \left(\sum_i a_i^2\right)\left(\sum_i b_i^2\right)$$

to get the proof of Mantel's theorem. Hint[1]

Solution: First, we get

$$\sum_{x \in V(G)} d(x)^2 = \sum_{xy \in E(G)} (d(x) + d(y)) \leq n|E(G)|$$

By Cauchy-Schwarz we have

$$\frac{1}{n}\left(\sum_{x \in V(G)} d(x)\right)^2 \leq \sum_{x \in V(G)} d(x)^2.$$

By the Handshaking lemma, the LHS is $\frac{1}{n}(2|E(G)|)^2$. Thus $\frac{1}{n}(2|E(G)|)^2 \leq n|E(G)|$. Solving for $|E(G)|$ gives the theorem.

[1] Handshaking lemma: $\sum_v d(v) = 2|E(G)|$

4: If G is triangle-free then the neighborhood of any vertex is an independent set. Let A be the largest independent set in G and let B be the remaining vertices. Thus $d(x) \leq |A|$. Use $\sum_{x \in B} d(x)$ and AGM inequality[2] to prove Mantel's theorem.

Solution: Every edge has an endpoint in B, thus by an application of the AGM inequality we get

$$|E(G)| \leq \sum_{x \in B} d(x) \leq |B||A| \leq \frac{(|B| + |A|)^2}{4} = \frac{n^2}{4}.$$

5: (Motzkin-Straus, 1965) To each vertex x assign a non-negative weight $w(x)$ such that $\sum_{x \in V(G)} w(x) = 1$. We would like to determine the maximum value of

$$S = \sum_{xy \in E(G)} w(x)w(y).$$

Assigning $1/n$ to each vertex gives that the maximum of S is $\geq |E(G)|/n^2$. Showing that S cannot exceed $1/4$ will complete the proof. We employ the "weight shifting technique." Let x and y be non-adjacent vertices and let W_x and W_y be the sum of the weights on vertices adjacent to x and y, respectively. Show that it is possible to shift weight from y to x. Then argue it is possible to shift all weight to just 2 vertices and thus prove the theorem.

Solution: Assume $W_x \geq W_y$ and let $\varepsilon \geq 0$. Thus

$$(w(x) + \varepsilon)W_x + (w(y) - \varepsilon)W_y \geq w(x)W_x + w(y)W_y.$$

This implies that we can shift all of the weight from one vertex y to some non-adjacent vertex x and not decrease S (if $W_y \leq W_x$). The graph G is triangle-free, so we can shift all of the weight to two adjacent vertices and not decrease S. Thus S is maximized at $1/4$ when these two vertices each have weight $1/2$.

[2] AGM states $4xy \leq (x+y)^2$. Comes from $\sqrt{xy} \leq \frac{x+y}{2}$ on geometric and arithmetic means.