More Turán's Theorem

Theorem 1 (Turán's Theorem, 1941)**.** *The maximum number of edges in an n-vertex graph with no $(k+1)$-clique is exactly $e(T_k(n))$. Furthermore, $T_k(n)$ is the unique graph attaining this maximum.*

Fifth proof. We begin with the following theorem about the independence number α of a graph.

Theorem 2 (Caro, 1979; Wei, 1981)**.** *Let G be a graph with independence number $\alpha(G)$, then*

$$\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{d(v)+1}.$$

Proof. (Alon-Spencer, 1992) Randomly order the vertices of G. The set of vertices that appear in this ordering before all of their neighbors forms an independent set S.

1: Calculate the expected size of S, which finishes the proof.

Solution: There are $\binom{n}{d(v)+1}$ ways to select locations for v and its neighbors in the permutation and $d(v)!$ ways to arrange those vertices (v must be first). The remaining elements can be arranged in $(n-d(v)-1)!$ ways. So the probability that a vertex v is in S in a given random ordering is

$$\frac{1}{n!}\binom{n}{d(v)+1} d(v)!(n-d(v)-1)! = \frac{1}{d(v)+1}.$$

Thus the expected size of S is simply the RHS of the inequality. In particular, there is an ordering of the vertices of G such that S is at least this large. The LHS is the maximum possible size of S.

□

Let H be an n-vertex graph.

2: Use Cauchy-Schwarz[1] on

$$n^2 = \left(\sum_{v\in V(H)} 1\right)^2 = \left(\sum_{v\in V(H)} \frac{\sqrt{d(v)+1}}{\sqrt{d(v)+1}}\right)^2 \leq \left(\sum_{v\in V(H)} (d(v)+1)\right)\left(\sum_{v\in V(H)} \frac{1}{d(v)+1}\right).$$

Then follow-up by using hand-shaking lemma to get $e(H)$ into the inequality. Finally, apply this to the complement of G, denoted by $\overline{G}$, i.e., use $\overline{G}$ as H. What can you tell about $\alpha(\overline{G})$? (Hint: Prove the Turán's theorem.)

Solution: Applying the Handshaking lemma and the theorem above we can replace the rightmost term with $(2e(H)+n)\alpha(H)$. Solving for $e(H)$ gives

$$e(H) \geq \frac{1}{2}\left(\frac{n^2}{\alpha(H)} - n\right). \tag{1}$$

[1] $(\sum a_i b_i)^2 \leq (\sum a_i^2)(\sum b_i^2)$

We are now ready to prove Turán's theorem. Given an n-vertex graph G with no $(k+1)$-clique consider its complement $\overline{G}$ which will contain no independent set of size $k+1$, i.e., $\alpha(\overline{G}) \le k$. Applying (1) to $\overline{G}$ gives $e(\overline{G}) \ge \frac{1}{2}(n^2/k - n)$. Thus

$$e(G) = \binom{n}{2} - e(\overline{G}) \le \binom{n}{2} - \frac{1}{2}\left(\frac{n^2}{k} - n\right) = \left(1 - \frac{1}{k}\right)\frac{n^2}{2}.$$

□

Sixth proof. (Li-Li, 1981; Kleitman-Lovász, 1994) Given a graph G we can assign a variable x_i to each vertex of G and define the polynomial

$$p_G = p_G(x_1, x_2, \ldots, x_n) = \prod_{i<j, ij \notin E(G)} (x_i - x_j).$$

An **identification** of a set of variables means that we set the variables equal to each other.

3: Show a key observation that a graph G is K_{k+1}-free if and only if for the identification of any set of $k+1$ variables, we get a polynomial $f_G = 0$.

Solution: If G contains K_{k+1}, then its identification does not give $f_G = 0$. If G is K_{k+1}-free, then any set of $k+1$ vertices contains a non-edge, this making $f_G = 0$.

Let $P(n)$ denote the set of n-variable polynomials where the identification of any set of $k+1$ variables gives the zero polynomial. It is easy to see that $P(n)$ forms an ideal in the ring of polynomials. Furthermore, let $\hat{P}(n)$ be the ideal generated by the polynomials $p_H(x_1, \ldots, x_n)$ where H is a k-partite n-vertex graph.

4: What is an ideal generated by some polynomials?

Solution:

$$\hat{P}(n) = \left\{ \sum_i p_{H_i} \cdot g_i \text{ where } g \text{ any polynom } \right\}$$

Now let G be an K_{k+1}-free n-vertex graph with the maximum number of edges. Clearly $p_G \in P(n)$. As in the previous proofs, it is enough to show that G is k-partite. We will need the following claim.

Claim 3. $P(n) = \hat{P}(n)$.

Proof. Clearly, $\hat{P}(n) \subset P(n)$, so let us show that $f \in P(n)$ implies $f \in \hat{P}(n)$. We proceed by induction on n. For $n = 2$ this is obvious so let $n > 2$ and assume the statement holds for smaller values. Let $S \subset [n-1]$ and define f_S as the polynomial resulting from replacing each x_i with x_n for $i \in S$ in the polynomial f. Now consider the polynomial

$$g = \sum_{S \subseteq [n-1]} (-1)^{|S|} f_S.$$

Which gives

$$f = g - \sum_{\emptyset \ne S \subseteq [n-1]} (-1)^{|S|} f_S.$$

If $S \ne \emptyset$, then f_S has at most $n-1$ variables, thus $f_S \in P(n-1)$ which by induction implies $f_S \in \hat{P}(n-1) \subset \hat{P}(n)$. Therefore to show that $f \in \hat{P}(n)$, it is enough to show that $g \in \hat{P}(n)$.

5: Show that x_i for $i < n$, then g is divisible by $(x_i - x_n)$.

Solution: If we replace x_i with x_n (for $i < n$) in g, then all terms cancel and we are left with the zero polynomial. Therefore g is divisible by $(x_1 - x_n)(x_2 - x_n)\cdots(x_{n-1} - x_n)$.

Hence we get

$$g = (x_1 - x_n)(x_2 - x_n)\cdots(x_{n-1} - x_n)h.$$

The identification of any $k+1$ variables in g gives $g = 0$, so if we identify any $k+1$ variables among $\{x_1, x_2, \ldots, x_{n-1}\}$ in h we must get $h = 0$. Factoring out terms of the form x_n^i in h gives

$$h = \sum h_i x_n^i$$

where each h_i is a polynomial over variables $x_1, x_2, \ldots, x_{n-1}$ and has the property that the identification of any of any $k+1$ of these variables gives $h_i = 0$, i.e., $h_i \in P(n-1) = \hat{P}(n-1)$.

Therefore, g can be written as the sum of polynomials of the form

$$x_n^i (x_1 - x_n)\cdots(x_{n-1} - x_n) p_{H'}$$

where $p_{H'}$ is the polynomial defined by the $(n-1)$-vertex k-partite graph H'.

6: How to change H' into a graph on n vertices? (and use some nice polynom)

Solution: Adding an isolated vertex (with variable x_n) to the graph H' gives an n-vertex graph H; Observe that $p_H = (x_1 - x_n)\cdots(x_{n-1} - x_n)p_{H'}$. Now one can add edges to make H' a complete k-partite, and it just removes some of the terms in the product.

Therefore, g can be written as the sum of polynomials of the form $x_n^i p_H$. Because $\hat{P}(n)$ is an ideal each of these terms is in $\hat{P}(n)$ and therefore $g \in \hat{P}(n)$. □

By the claim we have $p_G \in \hat{P}(n)$, so p_G can be written

$$p_G = \sum q_i p_{H_i}$$

where each where H_i is an n-vertex k-partite graph (and q_i are some other polynomials).

7: Show that $e(G) \le e(T_k(n))$ by considering the degrees of p_G and p_{H_i}.

Solution: Clearly, the degree of p_G is at least as large as that of each of the polynomials p_{H_i}, i.e.,

$$d(p_G) = \binom{n}{2} - e(G) \ge \binom{n}{2} - e(H) = d(p_{H_i})$$

Furthermore, $\deg(p_{H_i}) \ge \binom{n}{2} - e(T_k(n))$ as $T_k(n)$ is the n-vertex k-partite graph with the most edges. Rearranging terms gives that $e(G) \le e(T_k(n))$ completing the theorem.

□

Theorem 4 (Application of Mantel's theorem, Katona, 1969)**.** *Let u, v be independent and identically distributed random vectors in $\mathbb{R}^d$, then*

$$\mathbb{Pr}(|u+v| \geq 1) \geq \frac{1}{2}\mathbb{Pr}(|u| \geq 1)^2.$$

Proof. Suppose there are N vectors in the distribution and n of them have length at least 1. Then $\mathbb{Pr}(|u| \geq 1) = \frac{n}{N}$. Consider the graph with these n vectors as vertices and two (distinct) vectors u, v are connected by an edge if $|u+v| < 1$.

8: What can you say about the graph? (property and number of edges?)

Solution: It is easy to see that this graph will not contain triangles. Thus, by Mantel's theorem there are at most $\frac{n^2}{4}$ edges.

9: How many pairs of vectors u and v (not necessarily distinct) are there such that that $|u+v| \geq 1$? (use the graph)

Solution: The vectors u and v need not be distinct, so there are at least $\binom{n}{2}+n-\frac{n^2}{4} = \frac{n^2}{4}+\frac{n}{2}$ pairs of (distinct) vectors u, v such that $|u+v| \geq 1$.

10: Count the total number of pairs and finish the proof.

Solution: As u and v are not necessarily distinct, there are $\binom{N}{2}+N$ total possible pairs, u, v. This gives

$$\mathbb{Pr}(|u+v| \geq 1) \geq \frac{(1/4)n(n+2)}{(1/2)N(N+1)} \geq \frac{1}{2}\mathbb{Pr}(|u| \geq 1)^2.$$

$\square$