Goodman's bound and Moon-Moser

Instead of determining the maximum number of edges in a K_{k+1}-free graph we may ask how many copies of K_{k+1} are in a graph with some fixed number of edges. Moon and Moser gave a strong answer to this question that will lead to another proof of Turán's theorem.

As a warm-up, we start with an extension of Mantel's theorem. Let N_s be the number of copies of K_s in G.

Theorem 1 (Goodman bound)**.** *For every n-vertex graph G with m edges holds*

$$N_3 \geq \frac{m(4m-n^2)}{3n}.$$

The bound is not always tight. Tight asymptotic solution was obtained by Razborov and more precise count is in `https://arxiv.org/pdf/1712.00633.pdf`.

1: Show that Goodman bound is tight for Turán's graphs $T_k(\ell \cdot k)$.

Solution: Let $T_k(k\ell)$ be a Turán's graph on n vertices, i.e., $n = k\ell$. Thee vertices induce a triangle iff they are from three different parts, so $N_3 = \binom{k}{3}\ell^3$. On the other hand, $e = \binom{k}{2}\ell^2$, hence we get

$$\binom{k}{3}\ell^3 = N_3 = \frac{m(4m-n^2)}{3n} = \frac{\binom{k}{2}\ell^2(4\binom{k}{2}\ell^2 - (k\ell)^2)}{3k\ell}$$

2: Prove Goodman bound. Outline of the proof: For every edge xy, give a lower bound on the number of triangles containing xy (use $d(x), d(y), n$). Use the bound in $\sum$ over edges and change the $\sum$ to sum over vertices. And then use Cauchy-Schwartz[1].

Solution: The number of triangles using edge xy is at least $d(x) + d(y) - n$ (as this counts the number of common neighbors of x and y). Summing over all edges counts each triangle three times, so the total number of triangles is at least

$$\frac{1}{3}\sum_{xy\in E(G)}(d(x)+d(y)-n) = \frac{1}{3}\left(\sum_{x\in V(G)} d(x)^2 - nm\right).$$

Applying Cauchy-Schwartz inequality gives the total number of triangles is at least

$$\frac{1}{3}\left(\frac{1}{n}\left(\sum_{x\in V(G)} d(x)\right)^2 - nm\right) = \frac{4m}{3n}\left(m - \frac{n^2}{4}\right).$$

[1] $(\sum a_i b_i)^2 \leq (\sum a_i^2)(\sum b_i^2)$

Theorem 2 (Moon-Moser theorem)**.** *Let N_s be the number of copies of K_s in G. Then*

$$N_{s+1} \geq \frac{N_s}{s^2-1}\left(\frac{s^2 N_s}{N_{s-1}} - n\right).$$

Proof. Let S be a copy of K_s in G. Define $d(A)$ as the number of common neighbors of a set of vertices A. As a generalization of the Handshaking lemma we have

$$\sum_S d(S) = (s+1)N_{s+1}. \tag{1}$$

3: Why is (1) valid?

Solution: Double count pairs (S, x), where S is a copy of K_S and x is a vertex adjacent to all of S.

To prove Moon-Moser, we will count triples (S', x, y) such that S' is a copy of K_{s-1} and x and y are (not necessarily distinct) vertices each adjacent to all vertices of S'.

4: Count (S', x, y) by first picking S', use $d(S')$ for the calculation and use Cauchy-Schwarz and Handshaking (1) to give a lower bound.

Solution: Fixing, S', then x and y must be common neighbors of S', thus the total number of desired triples is

$$\sum_{S'} d(S')^2 \geq \frac{1}{N_{s-1}}\left(\sum_{S'} d(S')\right)^2 = \frac{s^2 N_s^2}{N_{s-1}}.$$

Where the first inequality is by Cauchy-Schwarz and the equality uses (1).

5: Count (S', x, y) by first considering xy. Make two cases - where you distinguish if xy is an edge or not (non-edge also works as $x = y$). For xy begin and edge, use N_{s+1}. For non-edge, what is $S' \cup \{x\}$ and use some bounds of how many choices are for y. The sum it all up!

Solution: If xy is an edge, then S' is inside a K_{s+1} and there are $(s+1)sN_{s+1}$ ways to form a desired triple. If xy is not an edge (allowing $x = y$), then $S' \cup \{x\}$ forms a K_s and y is adjacent to all vertices but x. Fix $S = K_s$ and pick a vertex y adjacent to all but one vertex of S (so y may also be inside of S), then we have a triple (S', x, y). There are at most $n - d(S)$ such choices for y. Thus the total number of triples is at most (using (1)).

$$(s+1)sN_{s+1} + \sum_S (n - d(S)) = (s+1)sN_{s+1} + nN_s - (s+1)N_{s+1} = (s^2-1)N_{s+1} + nN_s.$$

6: Combine the two estimates solving for N_{s+1}, which proves the theorem.

Solution: Obvious.

□

Corollary 3. *Let G is a graph with $e(G) = (1 - \frac{1}{x})\frac{n^2}{2}$. If N_s is the number of copies of K_{s+1} in G, then*

$$N_{s+1} \geq \left(1 - \frac{s}{x}\right)\frac{n}{s+1}N_s.$$

7: Prove the corollary by induction on s.

Solution:

Proof. Induction on s. For $s = 1$ the claim follows immediately as N_2 is the number of edges and N_1 is the number of vertices. Let $s > 1$ and assume the statement for smaller values. By the Moon-Moser theorem and the inductive hypothesis we have

$$N_{s+1} \geq \frac{N_s}{s^2-1}\left(\frac{s^2N_s}{N_{s-1}} - n\right) \geq \frac{N_s}{s^2-1}\left(\frac{s^2(1-\frac{s-1}{x})\frac{n}{s}N_{s-1}}{N_{s-1}} - n\right).$$

Simplifying the RHS yields the claim. □

The weak version of Turán's theorem follows easily.

8: Prove (weak) Turán's theorem using the corollary.

Solution:

Seventh proof of (weak) Turán's theorem. Let G be a graph with $e(G) > (1-\frac{1}{s})\frac{n^2}{2}$, and fix x such that $e(G) = (1-\frac{1}{x})\frac{n^2}{2}$ (note that $x > s$). We can repeatedly apply Corollary 3 to get that $N_3 > 0$, $N_4 > 0, \ldots, N_s > 0, N_{s+1} > 0$, i.e., G contains a copy of K_{s+1}. □

Another important corollary states that when G exceeds the number of edges given by Turán's theorem, then not only do we have a copy of K_{k+1} but in fact we have many copies. This property is called **supersaturation**.

Corollary 4. *Fix $\epsilon > 0$ and s, then there exists $c = c(\epsilon, s)$ such that any n-vertex graph G with $(1 - \frac{1}{s} + \epsilon)\frac{n^2}{2}$ edges, has at least cn^{s+1} copies of K_{s+1}.*

Proof. Set $\frac{1}{x} = \frac{1}{s} - \epsilon$, thus $x > s$.

9: Repeatedly apply Corollary 3 and get a lower bound on N_{s+1}.

Solution: Repeatedly applying Corollary 3 gives the number of copies of K_{s+1} is

$$N_{s+1} \geq N_1 \prod_{i=1}^{s}\left[\left(1 - \frac{i}{x}\right)\frac{n}{i+1}\right].$$

Simplifying gives

$$N_{s+1} \geq \left(\frac{n}{x}\right)^{s+1}\binom{x}{s+1} \geq \left(\frac{n}{x}\right)^{s+1}\left(\frac{x}{s+1}\right)^{s+1} = \left(\frac{n}{s+1}\right)^{s+1}.$$

□

Note that in the corollary above if we have that $\epsilon = 0$, then $s = x$ and the lower bound on N_{s+1} is simply 0.