Stability for F-free graphs

We now examine the structure of extremal and almost-extremal graphs. We will see that an F-free graph with close to the extremal number of edges closely resembles a Turán graph. This concept is called stability.

Theorem 1 (Füredi, 2015)**.** *Suppose G is an n-vertex K_{k+1}-free graph with $e(G) = e(T_k(n)) - t$. Then G can be obtained from a complete k-partite graph (on n vertices) by adding and removing at most $3t$ total edges.*

The theorem uses the following lemma.

Lemma 2 (Füredi, 2015)**.** *Suppose G is an n-vertex K_{k+1}-free graph with $e(G) = e(T_k(n)) - t$. Then G has a k-partite (i.e., k-colorable) subgraph G' with $e(G') \geq e(G) - t$.*

Proof. First we build a partition of G as follows: let x_1 be a vertex with maximal neighborhood $N_1 \subset G$ (i.e., x_1 has maximum degree) and let $V_1 = G - N_1$.

Now suppose that for for $j = 1, 2, \ldots, i-1$ we have defined x_j, N_j and V_j. Let x_i be a vertex with maximal neighborhood in the graph induced on N_{i-1}. Call this neighborhood N_i and put $V_i = N_{i-1} - N_i$.

1: Show the process stops (what graph is formed by x_is?) Are all vertices of G covered? Try to use $\sum_{i=1}^{k} |V_i||N_i|$ to somehow estimate $e(G)$ and also relate this to the number of edges in Turán graph. And hopefully get the proof out of it.

Solution:

Observe that $x_i \in N_{i-1} \subset N_{i-2} \subset \cdots \subset N_1$. Thus the x_is form a complete graph. Therefore, this process stops after at most k steps (for simplicity assume that it stops after choosing x_k), i.e., the vertex x_k has no neighbors in N_{k-1}. Thus $V_k = N_{k-1} - \emptyset$. The collection of subsets $V_1, V_2, \ldots, V_k$ form a vertex-partition of G.

First observe that the number of edges in the complete k-partite graph with classes $V_1, \ldots, V_k$ is

$$\sum_{i=1}^{k} |V_i||N_i|. \tag{1}$$

The Turán graph $T_k(n)$ has the most edges among the complete k-partite graphs, so (1) is a lower bound on $e(T_k(n)) = e(G) + t$.

On the other hand, for a vertex $x \in V_i$, the number of edges from x to a vertex in $V_i \cup V_{i+1} \cup \cdots \cup V_k$ is at most $|N_i|$. So (1) is an upper bound for $e(G) + \sum_i e(V_i)$ (as edges inside of V_i are counted twice).

Combining the two bounds for (1) and simplifying gives $\sum_i e(V_i) \leq t$, i.e., after removing the (at most) t edges inside the V_is we are left with a k-partite graph G'

$\square$

Theorem 3 (First stability theorem)**.** *For any $\epsilon > 0$ and any $(k+1)$-chromatic graph F, there exists $\delta > 0$ and n_0 such that if G is an F-free n-vertex graph with $n > n_0$ and*

$$e(G) > \left(1 - \frac{1}{k}\right)\frac{n^2}{2} - \delta n^2,$$

then G can be obtained from $T_k(n)$ by adding and removing at most ϵn^2 total edges.

The first stability theorem essentially states that F-free graphs that have close to the maximal number of edges closely resemble the Turán graph.

Theorem 4 (Asymptotic structure theorem)**.** *If G is an n-vertex extremal graph for a $(k+1)$-chromatic graph F, then the minimum degree of G is*

$$\delta(G) = \left(1 - \frac{1}{k} + o(1)\right)n.$$

Proof. By the first stability theorem we know that $T_k(n)$ can be transformed into G by adding and removing $o(n^2)$ total edges. For some transformation and some vertex v, let $r(v)$ be the number of edges incident to v that were removed.

Let $V_1, V_2, \ldots, V_k$ be the partition classes of $T_k(n)$ and let V_1 be a class of minimal size, i.e., $|V_1| = \lfloor n/k \rfloor \leq n/k$.

2: Finish the proof. Idea: There are not many edges removed incident with V_1, so we could use some symmetrization technique (Zykov's proof of Turán's theorem - all vertices have same degree) - but the symmetrization needs to something bigger than just a vertex (preserve F-free).

Solution: By the first stability theorem we know that the number of edges incident to V_1 removed in the transformation is

$$\sum_{v \in V_1} r(v) = o(n^2)$$

Now consider a set S of $|V(F)|$ many vertices in V_1 such that $\sum_{v \in S} r(v)$ is minimal. Thus by averaging, we have

$$\sum_{v \in S} r(v) \leq \frac{|S|}{|V_1|} \sum_{v \in V_1} r(v) = o(n).$$

We now use the symmetrization technique from Zykov's proof of Turán's theorem. Let v be an arbitrary vertex of G. Let $d(S)$ be the number of vertices in the common neighborhood of S. If $d(v) < d(S)$, then remove all edges incident to v and connect v to all the common neighbors of S. Observe that the resulting graph remains F-free and has more edges; a contradiction. Thus,

$$d(v) \geq d(S) \geq n - \left\lfloor \frac{n}{k} \right\rfloor - \sum_{v \in S} r(s) \geq n - \frac{n}{k} - o(n).$$

□

Supersaturation

A consequence of the Moon-Moser theorem was that if we have an n-vertex graph with $> \mathrm{ex}(n, K_{k+1})$ edges, then we not only have a copy of K_{k+1} but cn^{k+1} many copies. We generalize this result for all graphs F.

Lemma 5. *Fix q and ϵ positive reals and suppose G is an n-vertex graph with $e(G) > (q+\epsilon)\binom{n}{2}$ many edges. Then for $m \geq 2$, there exists $\delta > 0$ such that G contains at least $\delta\binom{n}{m}$ many induced subgraphs on m vertices with at least $(q+\epsilon/2)\binom{m}{2}$ many edges.*

Proof. **3:** Suppose for contradiction the lemme is false and double-count the number of pairs (S, e), where e is an edge and S is a vertex set of size m containing e. Show that is δ is really small, contradiction happens.

Solution: Let $e(S)$ count the number of edges in the induced graph on vertices $S \subset V(G)$. Each edge of G is contained in $\binom{n-2}{m-2}$ many vertex sets of size m, so

$$\sum_{S \in \binom{V(G)}{m}} e(S) = \binom{n-2}{m-2}(q+\epsilon)\binom{n}{2}.$$

On the other hand, if we suppose that the lemma is false, then all but $\delta\binom{n}{m}$ many m vertex sets have $< (q+\epsilon/2)\binom{m}{2}$ edges and the remaning have at most $\binom{m}{2}$ edges. So

$$\sum_{S \in \binom{V(G)}{m}} e(S) < \binom{n}{m}\binom{m}{2}\left(q+\frac{\epsilon}{2}\right) + \delta\binom{n}{m}\binom{m}{2} < \binom{n}{m}\binom{m}{2}\left(q+\frac{3\epsilon}{4}\right)$$

for δ chosen to be small enough. Combining these two estimates gives

$$\binom{n-2}{m-2}\binom{n}{2}(q+\epsilon) < \binom{n}{m}\binom{m}{2}\left(q+\frac{3\epsilon}{4}\right).$$

This is a contradiction as $\binom{n-2}{m-2}\binom{n}{2} = \binom{n}{m}\binom{m}{2}$.

□

Theorem 6. *Fix a graph F on f many vertices. For $\epsilon > 0$, there exists an $\alpha > 0$ such that if G is an n-vertex graph with*

$$e(G) > \mathrm{ex}(n, F) + \epsilon n^2,$$

then G contains at least αn^f many copies of F. (for n large enough)

Proof. **4:** Use Erdős-Stone-Simonovits to get q and then use the lemma above but! use it for real large n and so you can use the dense pieces to harvest copies of F. And then fix possible overcounting.

Solution: By ESS, we know that

$$\mathrm{ex}(n, F) \leq \left(1 - \frac{1}{\chi(F)-1}\right)\binom{n}{2} + o(n^2).$$

Put $q = 1 - \frac{1}{\chi(F)-1}$ and fix m large enough such that

$$\mathrm{ex}(m, F) < \left(q + \frac{\epsilon}{2}\right)\binom{m}{2}.$$

Now apply the above lemma with to find $\delta\binom{n}{m}$ many vertex subsets S of size m such that

$$e(S) \geq \left(q + \binom{\epsilon}{2}\right)\frac{m}{2} > \mathrm{ex}(n, F).$$

Each such S clearly must contain a copy of F for a total of $\delta\binom{n}{m}$ many copies of F. However, these copies may have been counted many times. A fixed copy of F appears in $\binom{n-f}{m-f}$ many vertex sets of size m, so the number of copies of F is at least

$$\frac{\delta\binom{n}{m}}{\binom{n-f}{m-f}} \geq \alpha n^f$$

for some β.

$\square$