C_4-free graphs

Lemma 1 (Jensen's inequality)**.** *If $0 \le \lambda_i \le 1$, $\sum \lambda_i = 1$ and f is a convex function, then*

$$f\left(\sum \lambda_i x_i\right) \le \sum \lambda_i f(x_i).$$

If each $\lambda_i = \frac{1}{n}$ (and the sum has n terms), then Jensen's inequality simplifies to

$$f\left(\frac{1}{n}\sum x_i\right) \le \frac{1}{n}\sum f(x_i).$$

For graphs of chromatic number 3 or greater, ESS gives a good answer for the Turán number. Our only hope is to improve the smaller-order terms. However, when H is bipartite, then ESS only says that $\mathrm{ex}(n, H) = o(n^2)$ so it is a natural question to try to improve this.

Theorem 2 (Erdős, 1938)**.**

$$\mathrm{ex}(n, C_4) \le \frac{n}{4}\left(\sqrt{4n-3}+1\right) \sim \frac{1}{2}n^{3/2}.$$

Proof. Let us count the number of paths on three vertices by their center vertex (it is helpful to think of the two edges forming a letter V or as a $K_{1,2}$).

1: Give an upper bound on the number of V's using C_4-free property and give lower bound using Jensen's inequality.

Solution: The number of Vs centered at x is $\binom{d(x)}{2}$. Therefore the total number of Vs is

$$\sum_{v \in V(G)} \binom{d(v)}{2} \ge n\binom{\frac{1}{n}\sum d(v)}{2} = n\binom{2e(G)/n}{2} = \frac{2e(G)^2}{n} - e(G).$$

Where the first inequality is due to Jensen's inequality (as $\binom{x}{2}$ is convex) and the first equality uses the Handshaking lemma.

On the other hand, two Vs cannot share the same endpoints as this would form a C_4, so there are at most $\binom{n}{2}$ total Vs. Combining these bounds gives $\binom{n}{2} \ge \frac{2e(G)^2}{n} - e(G)$. Solving for $e(G)$ (use the quadratic formula) gives the theorem.

□

The lower bound on $\mathrm{ex}(n, C_4)$ was given by Erdős-Rényi-Sós and involves a bit of number theory and gives the full theorem.

Theorem 3.

$$\mathrm{ex}(n, C_4) \sim \frac{1}{2}n^{3/2}.$$

Proof. The upper bound is the previous theorem. For the lower bound we will construct a C_4-free graph G with the desired number of edges.

Let p be prime and define a graph G with vertex set $\mathbb{F}_p \times \mathbb{F}_p - \{0, 0\}$ where distinct pairs (x, y) and (a, b) form an edge iff $ax + by = 1$.

2: What is n? What is the (minimum) degree of G? What is then the lower bound on the number of edges? Is G actually C_4-free?

Solution: Observe that $n = p^2 - 1$. For a fixed (x, y) it is easy to check that there are p solutions (a, b) to $ax + by = 1$ (when $x = 0$, then a anything, but $b = y^{-1}$ and when x and y both non-zero, a can be anything, but then $b = (1 - ax)y^{-1}$). It is possible that one solution is (x, y) itself which we discard. Therefore, the degree of (x, y) is at least $p - 1$ in G. Thus, $e(G) \geq \frac{1}{2}(p-1)n = \frac{1}{2}(p-1)(p^2-1) \sim \frac{1}{2}p^3 \sim \frac{1}{2}n^{3/2}$. Now let us confirm that G is C_4-free. If two vertices (a, b), and (a', b') are non-adjacent in a C_4, then there would be two solutions (x, y) to the equations $ax + by = 1$ and $a'x + b'y = 1$ which is clearly impossible.

Finally, note that this construction only works if $\sqrt{n+1}$ is prime. However, for any n, there is a prime between $(1 - o(1))\sqrt{n}$ and $\sqrt{n}$ which will still result in a construction (just add some isolated vertices to $p^2 - 1$ to get n) with leading term $\sim \frac{1}{2}n^{3/2}$.

$\square$

The graph C_4 is both a cycle and a complete bipartite graph. We will first generalize the above result to complete bipartite graphs

Theorem 4 (Erdős; Kővari-Sós-Turán, 1954). *For any naturals $s \leq t$ we have*

$$\mathrm{ex}(n, K_{s,t}) \leq \frac{1}{2}(t-1)^{1/s} n^{2-1/s} + O(n).$$

The proof is simply a generalization of the proof for C_4.

3: Let G be an n-vertex graph with no $K_{s,t}$ subgraph. Count the pairs (v, S) where v is a vertex and S is a set of s vertices in the neighborhood of v. Use this to finish the proof.