Applications of C_4-free graphs

We begin with the problem that motivated the study of the Turán number of the C_4.

Theorem 1 (Erdős[1], 1938). *Let $A \subset [n]$ be a set of integers such that all products of pairs of elements of A are distinct. Then*

$$|A| \leq \pi(n) + O(n^{3/4})$$

where $\pi(n)$ is the number of primes not exceeding n (recall $\pi(n) \approx n/\ln n$).

Note that the set of primes form a construction of A of size $\pi(n)$. To prove the theorem we need two easy lemmas.

Lemma 2. *Let D be the naturals at most $n^{2/3}$ and B be the set containing D and all primes between $n^{2/3}$ and n. Then each integer in $[n]$ can be written as a product of an element of D and B.*

Proof. **1:** Prove Lemma 2.

Solution: We show that $x \in [n]$ can be written as the product $d \cdot b$ such that $d \in D$ and $b \in B$. If $x \leq n^{2/3}$, then $x = 1 \cdot x$ is a solution, so assume $x > n^{2/3}$. If x has a prime factor p that is greater than $n^{2/3}$ then $x = p \cdot (x/p)$ is a solution as $(x/p) \leq n^{2/3}$, so we can assume all prime factors are at most $n^{2/3}$. Then some product q of prime factors of x is between $n^{1/3}$ and $n^{2/3}$, thus $x = q \cdot (x/q)$ is a solution.

□

Lemma 3. *Let G be a C_4-free bipartite graph with class sizes a and b, then*

$$e(G) \leq a\sqrt{b} + b.$$

2: Prove Lemma 3.

Hint: Similar to the proof of KST (count cherries centered in B, recall Jensen)

Solution: Let G be a C_4-free bipartite graph with classes of size a and b. Call the classes A and B, respectively Let us count the number of Vs with center in B. Each pair of vertices A has at most one common neighbor in B so the number of such Vs is at most

$$\binom{a}{2} \leq \frac{a^2}{2}.$$

On the other hand, the number of Vs on a vertex x in B is $\binom{d(x)}{2}$. So the total number of such Vs is

$$\sum_{x \in B} \binom{d(x)}{2} \geq b \binom{\frac{1}{b} \sum d(v)}{2} = b \binom{e(G)/b}{2} \geq b \frac{(e(G)/b - 1)^2}{2}.$$

Combining these two estimates for the number of Vs and solving for $e(G)$ completes the proof.

[1] Erdős showed that the error term is between $O\left(\frac{n^{3/4}}{(\log n)^{3/2}}\right)$ and $O(n^{3/4})$.

Proof of Theorem 1, up to error term. Let A be a set as in the theorem and consider each element of A as its product given by Lemma 2. There may be many such representations, so choose the representation where the element of B is smallest. Consider the bipartite graph with vertex set $B \cup D$ and connect two vertices $b \in B$ and $d \in D$ if there is an element in a with representation $b \cdot d$.

3: Show that G is C_4-free.

Solution: Suppose for contradiction that there is a C_4 in G. Denote its vertices by $b_1, b_2 \in B$ and $d_1, d_2 \in D$. Notice that $b_1 d_1 \cdot b_2 d_2 = b_1 d_2 \cdot b_2 d_1$, while all four products are in A, which is a contradiction.

We need to be more careful when applying the edge-bound.

We partition the elements $x \in B$ into three parts B_1, B_2, B_3 such that B_1 contains $x < n^{1/2}$, the set B_2 contains $n^{1/2} \leq x < n^{3/4}$, and B_3 contains the rest.

4: Notice what are adjacencies in D for each of the B_is and finish the proof by applying Lemma 3 a couple times.

Solution: Observe that B_2 is only adjacent to vertices in D that are at most $n^{1/2}$ and B_3 is only adjacent to vertices in D that are at most $n^{1/3}$

Applying Lemma 3 to each class of B and its potential neighbors gives

$$e(G) = \pi(n) + O(n^{5/6}) = (1 + o(1))\pi(n).$$

□

For the next application, we restate KST theorem.

Theorem 4 (Erdős; Kővari-Sós-Turán, 1954)**.** *For any naturals $s \leq t$ we have*

$$\mathrm{ex}(n, K_{s,t}) \leq \frac{1}{2}(t-1)^{1/s} n^{2-1/s} + O(n).$$

To get the next application we need a definition. Given a set A, the **sumset** of A is the set of all sums of pairs of elements of A; denoted $A + A = \{a + a' \mid a, a' \in A\}$. We are interested in determining the minimum possible size of a set $A \subset \mathbb{N}$ such that $A + A$ contains all perfect squares $1^2, 2^2, 3^2, \ldots, n^2$.

Theorem 5 (Erdős-Newman[2], 1977)**.** *Given $\epsilon > 0$ and n large enough, if A is a set of non-negative integers such that $\{1^2, 2^2, 3^2, \ldots, n^2\} \subset A + A$, then $|A| \geq n^{2/3-\epsilon}$.*

Proof. Let G be a graph with vertex set A and two vertices a and a' are connected by an edge if $a + a'$ is a perfect square at most n^2. Observe that G has at least n edges.

5: Estimate the number of common neighbors of two vertices. Use KST on $K_{2,t}$-free graphs for a suitable t. Useful to know: If $d(x)$ is the number of divisors of x, then a consequence of the prime number theorem is that for any $\delta > 0$, that $d(x) = o(x^\delta)$.

Solution: Let us determine an upper-bound on the number of common neighbors b of two fixed vertices a and a'. Then there exists x^2 and y^2 such that $a + b = x^2$ and $a' + b = y^2$ which implies $a - a' = x^2 - y^2$. Therefore $x - y$ and $x + y$ divide $a - a'$.

[2]The best known construction has $\frac{n}{\log^{\omega(1)} n}$ elements.

Each common neighbor b uniquely determines x and y, thus there are at most as many common neighbors as pairs of divisors of $a - a'$. If $d(x)$ is the number of divisors of x, then we have that the number of common neighbors is at most $(d(a - a'))^2$. It is well-known (e.g., it is a consequence of the prime number theorem.) that for any $\delta > 0$, that $d(x) = o(x^\delta)$. Note that $a - a' \leq n^2$, so for n large enough we can force $(d(a-a'))^2 = o(n^{3\epsilon})$ for all pairs a, a'. Thus, for n large enough, the number of common neighbors of a pair of vertices is less than $n^{3\epsilon}$. So for any $t = n^{3\epsilon}$ we have that G is $K_{2,t}$-free. By KST we get that

$$n \leq e(G) \leq (n^{3\epsilon})^{1/2}|A|^{3/2}.$$

Solving for $|A|$ gives the theorem.

□

We only have sharp (in the order of magnitude) constructions for some specific values of s and t. In general, we have

Theorem 6. *For naturals $s \geq t$ we have*

$$\mathrm{ex}(n, K_{s,t}) \geq \frac{1}{16} n^{2-\frac{s+t-2}{st-1}}.$$

Proof. Build a graph randomly with edge probability

$$p = \frac{1}{2} n^{-\frac{s+t-2}{st-1}}.$$

6: Finish the proof.

Solution: The expected number of edges is $p\binom{n}{2} \geq \frac{1}{4}pn^2$. The expected number of copies of $K_{s,t}$ is at most

$$\binom{n}{s}\binom{n}{t}p^{st} \leq n^{s+t}p^{st}.$$

Let X be the random variable defined by the difference between the number of edges and number of $K_{s,t}$s. By linearity of expectation we have that $E(X) \geq \frac{1}{4}pn^2 - n^{s+t}p^{st}$. By the definition of expectation, there exists an n-vertex graph such that $X \geq E(X)$. This means that if we remove one edge from every $K_{s,t}$ we are left with an n-vertex graph G that is $K_{s,t}$-free and has

$$e(G) \geq \frac{1}{4}pn^2 - n^{s+t}p^{st} \geq \frac{1}{8}pn^2 = \frac{1}{16}n^{2-\frac{s+t-2}{st-1}}.$$

□