Dependent Random Choice

We would like to prove a general upper-bound on $\mathrm{ex}(n, F)$ where F is a bipartite graph. First we need a lemma that has many applications.

Rough version of the Lemma: In every somewhat dense graph G exists a subset of vertices A, such that every subset of k vertices in A has at least m common neighbors.

Lemma 1 (Dependent random choice)**.** *Fix positive integers k, m, a and let G be an n-vertex graph with average degree $d = 2e(G)/n$. If there is a positive integer t such that*

$$\frac{d^t}{n^{t-1}} - \binom{n}{k}\left(\frac{m}{n}\right)^t \geq a, \tag{1}$$

then G contains a subset A of a vertices such that every set of k vertices in A has at least m common neighbors.

Notes and proof idea: It is not a good idea to take a random set A, for example taking G as a disjoint union of two cliques will likely produce A, where many pairs of vertices have no common neighbors. Actual proof idea is to randomly pick t vertices and consider the common neighborhood of these t vertices, it would be almost A, call it A' and obtain the real A by removing vertices to get rid of all subsets of size k that do not have m common neighbors. If a set S has small neighborhood, then it is unlikely to be in A' in the first place. So one does not have to remove too much to get rid of them.

Proof. Pick a set of t vertices $b_1, b_2, \ldots, b_t$ uniformly at random *with repetition*. Define X to be the random variable measuring the size of the common neighborhood of $b_1, \ldots, b_t$.

1: Calculate $\mathbb{E}[X]$. Hints: Indicator for each vertex to be in the neighborhood and Jensen. See positive in (1).

Solution: The probability that a vertex v is in the common neighborhood of $b_1, \ldots, b_t$ is simply the probability that the b_is are neighbors of v, so

$$\mathbb{P}\mathrm{r}[b_1, \ldots, b_t \in N(v)] = \left(\frac{d(v)}{n}\right)^t.$$

Therefore, the expectation of X is

$$\mathbb{E}[X] = \sum_{v \in V(G)} \left(\frac{d(v)}{n}\right)^t = \frac{1}{n^t} n \left(\frac{\sum_{v\in V(G)} d(v)^t}{n}\right) \geq n^{1-t}\left(\frac{\sum_{v\in V(G)} d(v)}{n}\right)^t = \frac{d^t}{n^{t-1}}$$

by Jensen's inequality on the convex function x^k.

We call a set of k vertices "bad" if it has fewer than m common neighbors. Again, for $b_1, \ldots, b_t$ chosen randomly, let Y be the random variable counting the number of bad k-sets in the common neighborhood of $b_1, \ldots, b_t$.

2: Calculate $\mathbb{E}[Y]$. Look at negative term in (1).

Solution: A fixed bad set R has fewer than m common neighbors. If the common neighborhood of R contains the b_i, then R is in the common neighborhood of the $b_i s$. Thus, the probability that R is in the common neighborhood of the b_is is less than $(\frac{m}{n})^t$.

There are at most $\binom{n}{k}$ bad sets, so

$$\mathbb{E}[Y] < \binom{n}{k}\left(\frac{m}{n}\right)^t.$$

By linearity of expectation we have

$$\mathbb{E}[X - Y] \geq \frac{d^t}{n^{t-1}} - \binom{n}{k}\left(\frac{m}{n}\right)^t \geq a.$$

Therfore, by the proablistic method, there exists a set of vertices A' such that $X - Y \geq a$. Deleting one vertex from each bad k-set in A' gives a set A with at least a vertices and all k-sets have at least m common neighbors.

□

Once we have found a set A with the desired properties, we can find many different graphs as a subgraph.

Theorem 2. *Let F be a bipartite graph with maximum degree Δ, then there is a constant c (depending on F) such that*

$$\mathrm{ex}(n, F) \leq cn^{2-1/\Delta}.$$

Proof. Suppose F has classes of size a and b and let G be a graph with n vertices and at least $cn^{2-1/\Delta}$ edges (where c is to be determined later). The average degree of G is $d \geq 2cn^{1-1/\Delta}$.

3: Use dependent random choice lemma to show that G contains a set of size a such that every subset of size Δ has at least $a + b$ common neighbors. Hint: Use $t = \Delta$, you can choose c nicely.

Solution: We apply the dependent random choice lemma with $k = \Delta$, $m = a + b$ and a. Put $t = \Delta$ and observe that

$$\frac{d^t}{n^{t-1}} - \binom{n}{k}\left(\frac{m}{n}\right)^t = \frac{d^\Delta}{n^{\Delta-1}} - \binom{n}{\Delta}\left(\frac{a+b}{n}\right)^\Delta \geq (2c)^\Delta - n^\Delta\left(\frac{a+b}{n}\right)^\Delta = (2c)^\Delta - (a+b)^\Delta.$$

So choose c such that the RHS is at least a. Therefore, G has a vertex subset A of size a such that all subsets of A of size $k = \Delta$ have at least $m = a + b$ common neighbors.

4: Now we show that F can be embedded into the graph G, i.e., F is a subgraph of G.

Solution: Embed the vertices in the class of size a of F into A arbitrarily. Now suppose we have already embedded into G some of the vertices of H in the class of size b. Let v be a vertex that has not yet been embedded. The vertex v is adjacent to at most Δ vertices in F. These vertices have already been embedded into A. Therefore they have at least $a + b$ common neighbors in G. Less than $a + b$ vertices of F have already been embedded into G, so we can embed v into one of the remaining common neighbors. We can embed all vertices this way to get F as a subgraph of G.

□

The 1-**subdivision** of a graph G is the graph resulting from subdividing each edge of G. That is, each edge of G is split into two edges by added a vertex in the middle of the edge.

Theorem 3. *If G is a graph with n vertices and ϵn^2 edges, then G contains a 1-subdivision of the complete graph K_a on $a = \epsilon^{3/2} n^{1/2}$ vertices.*

Proof. Let H be the subdivision of K_a and observe that H is a bipartite graph with classes of size a and $\binom{a}{2}$ where each pair of vertices in the class of size a has a unique common neighbor in the class of size $\binom{a}{2}$.

5: Use dependent random choice to show G contains a set A of a vertices such subset of A of size ? has at least ? common neighbors. What should be the ?... see next exercise.
Hints: Later use $m \le a^2$ and $t = \frac{\log n}{2 \log 1/\epsilon} = \log_\epsilon n^{-1/2}$.

Solution: The average degree of G is $d = 2\epsilon n$. We would like to apply dependent random choice with $k = 2$ and $m = a + \binom{a}{2} \le a^2$. So consider

$$\frac{d^t}{n^{t-1}} - \binom{n}{k}\left(\frac{m}{n}\right)^t \ge (2\epsilon)^t n - \frac{n^2}{2}\epsilon^{3t} = 2^t \epsilon^t n - \frac{n^2}{2}\epsilon^{3t}$$

If we pick $t = \frac{\log n}{-2 \log \epsilon} = \log_\epsilon n^{-1/2}$, then the RHS is

$$2^t \epsilon^t n - \frac{n^2}{2}\epsilon^{3t} = 2^t n^{1/2} - \frac{n^{1/2}}{2} \ge n^{1/2} \ge a.$$

Therefore, we have a vertex set A of a vertices such that every subsets of $k = 2$ vertices has at least $a + \binom{a}{2}$ common neighbors.

6: Find an embedding of 1-subdivision of K_a using A.

Solution: As in the previous proof we can embed H easily. Embed the vertices in the class of size a of H into A arbitrarily. Now suppose we have already embedded into G some of the vertices of of H in the class of size $\binom{a}{2}$. Let v be a vertex that has not yet been embedded. The vertex v is adjacent to at most $k = 2$ vertices in H. These vertices have already been embedded into A. Therefore they have at least $a + \binom{a}{2}$ common neighbors in G. Less than $a + \binom{a}{2}$ vertices of H have already been embedded into G, so we can embed v into one of the remaining common neighbors. We can embed all vertices this way to get H as a subgraph of G.

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