Paths and Matchings

Theorem 1 (Erdős -Gallai[1], 1959).

$$\mathrm{ex}(n, P_{k+1}) \leq \frac{n}{k}\binom{k}{2} = \frac{k-1}{2}n.$$

Note we think of P_k as a path on k vertices.

1: Show that the bound given by Erdős-Gallai is sharp when k divides n, i.e., find a construction.

Solution: Consider the graph of n/k copies of K_k which clearly does not contain P_{k+1} and has the desired number of edges. Before proving the theorem we will need an intermediate lemma that will be of interest later.

Lemma 2. *If G is a connected n-vertex graph with minimum degree $\delta(G)$, then G contains a path on $\min\{n, 2\delta(G)+1\}$ vertices.*

Proof. Let P be a path of maximum length in G with endpoints x and y. Clearly all neighbors of x and y are inside of P (otherwise we get a longer path). If P contains all n vertices of G, then we are done, so we may assume that there is a vertex z not on P but incident to some internal vertex of P (as G is connected).

Let S be the set of vertices immediately preceding the neighbors of x and let $N(y)$ be the neighbors of y.

2: Use the pigeonhole principle to finish the proof.

Solution: Clearly these two sets are subsets of the vertices of $P - y$ and each is of size $\delta(G)$. Thus, by the pigeonhole principle, if $|P - y| < 2\delta(G)$ we get $S \cap N(y) \neq \emptyset$ which implies that P contains the following configuration (note that the right figure is a special case of the left).

In both cases we can form a longer path with z. Thus, $|P - y| \geq 2\delta(G)$ and thus P must contain at least $2\delta(G) + 1$ vertices.

□

Proof of Erdős-Gallai. **3:** Use induction on n (vertex of small degree) and Lemma 2 to finish the proof.

Solution: Induction on n. For $n \leq k + 1$, the theorem is trivial so let $n > k + 1$ and assume the theorem holds for smaller graphs. If G is not connected, then we can apply induction to each component to get the theorem so we will assume that G is connected. By Lemma 2 we may assume that G contains a vertex of degree less than $\frac{k}{2}$ (otherwise we get a path on $k + 1$ vertices). However, if we remove this vertex and apply induction to the remaining graph we get the theorem.

□

[1] The exact value of $\mathrm{ex}(n, P_k)$ for all values of n was found by Faudree and Schelp

We can also consider the Turán number of graphs that are not connected. Hall's theorem will be helpful.

4: State Hall's theorem

Solution:

Theorem 3 (Hall's marriage theorem, 1935)**.** *Let A and B be the classes of a bipartite graph. There is a matching incident to each vertex of A if and only if for every $S \subset A$ we have $|S| \le |N(S)|$.*

Theorem 4 (Erdős-Gallai[2], 1959)**.** *Let $k \cdot K_2$ be the graph of k independent edges, then*

$$\mathrm{ex}(n, k\cdot K_2) = \max\left\{\binom{k-1}{2} + (k-1)(n-k+1), \binom{2k-1}{2}\right\}.$$

Proof (adapted from Brandt). We prove the upper bound by contrapositive, i.e., if an n-vertex graph G has $e(G) > \mathrm{ex}(n, F)$ then G contains k independent edges. We proceed by induction on k. Clearly if $k = 1$ the theorem holds, so let $k > 1$ and assume the statement holds for smaller values.

First we consider the case when G has minimum degree k. Let A be a set of k vertices such that the number of edges in A is minimal; let B be $G - A$. Consider the bipartite graph between A and B.

5: Use Hall's theorem to show that there is a matching between A and B, which finishes the proof of the theorem.

Solution: Suppose (to the contrary) that there is no matching incident to each vertex of A, then by Hall's theorem, there is a set $X \subset A$ such that $|X| > |N(X)|$. Let x be an arbitrary vertex of X. Clearly x has at most $|N(X)| < |X|$ neighbors in B, thus x has more than $k - |X|$ neighbors in A (as the degree of x is at least k). However, if y is an arbitrary vertex of $B - N(X)$, then y is not adjacent to X and therefore has at most $k - |X|$ neighbors in A. Thus swapping x and y in A contradicts the construction of A. Therefore, G contains a k-matching.

Now we may assume G contains a vertex v of degree at most $k-1$. Let u be an arbitrary neighbor of v.

6: Consider the graph $G - v - u$ and finish the proof by using induction.

Solution: Then

$$e(G-v-u) \ge e(G) - (k-1) - (n-2) > \binom{k-1}{2} + (k-1)(n-k+1) - (k-1) - (n-2).$$

With some calculation (two cases) we can show

$$e(G-v-u) > \mathrm{ex}(n-2, (k-1)\cdot K_2).$$

So by induction $G - u - v$ contains a $(k-1)$ independent edges. Together with uv this gives a k independent edges.

7: Find a construction that shows the Theorem is tight.

[2] Brandt showed that the upper-bound holds for any forest on k edges (without isolated edges)

Solution:

□