Combinatorial Counting - 3.4 - 3.6 Estimates II

How quickly H_n grows? The answer is something like $\log_2 n$. But it is not exactly that.

This motivates "Big-O" notation.

Definition: Let f, g be functions $\mathbb{N} \to \mathbb{R}$. If there exists a constant $C > 0$ such that $|f(n)| \leq C \cdot g(n)$ for all n, then we denote it by $f(n) = O(g(n))$ or just $f = O(g)$.

Note: C can be quite large! Sometimes defined that $|f(n)| \leq C \cdot g(n)$ for n sufficiently large.

Rules: If $f_1 = O(g_1)$ and $f_2 = O(g_2)$ then

- $f_1 + f_2 = O(g_1 + g_2)$
- $f_1 \cdot f_2 = O(g_1 \cdot g_2)$

1: Use the rules to show $(n^2 + \log(n)) \cdot (14n^3 + 2n^2 + \sqrt{n}) = O(n^5)$.

Solution: We use $n^2 + \log(n) = O(n^2 + n^2) = O(n^2)$ and $14n^3 + 2n^2 + \sqrt{n} = O(n^3 + n^3 + n^3) = O(n^3)$. The the product is $O(n^2 \cdot n^3) = O(n^5)$.

2: Prove that the rules are correct.

Solution: Use from definition

Useful estimates

- $n^\alpha = O(n^\beta)$ if $\alpha \leq \beta$
- $n^C = O(\alpha^n)$ for any C and $\alpha > 1$
- $(\ln n)^C = O(n^\alpha)$ for any C and $\alpha > 0$.

Other notation

Notation	Definition	Meaning
$f(n) = o(g(n))$	$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$	f grows way slower than g
$f(n) = \Omega(g(n))$	$g(n) = O(f(n))$	f grows at least as fast as g
$f(n) = \Theta(g(n))$	$f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$	f and g have roughly similar growth
$f(n) \sim g(n)$	$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$	f and g are almost the same

Simple estimate for $n!$:

$$n^{n/2} \leq n! \leq \left(\frac{n+1}{2}\right)^n$$

Better estimate

$$e\left(\frac{n}{e}\right)^n \leq n! \leq en\left(\frac{n}{e}\right)^n$$

Almost true

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

3: Show that $n! \leq en\left(\frac{n}{e}\right)^n$ using induction on n for $n \geq 1$. Hint: Use $1 + x \leq e^x$ for all $x \in \mathbb{R}$.

Solution: Base case: $n = 1$ holds.

Now by induction

$$\begin{aligned} n! = n(n-1)! = n \cdot e(n-1)\left(\frac{n-1}{e}\right)^{n-1} &= n \cdot e(n-1)\left(\frac{n}{n}\right)^n\left(\frac{n-1}{e}\right)^{n-1} \\ &= n \cdot e^2\left(\frac{n}{e}\right)^n\left(\frac{n-1}{n}\right)^n = n \cdot e^2\left(\frac{n}{e}\right)^n\left(1 - \frac{1}{n}\right)^n \leq n \cdot e^2\left(\frac{n}{e}\right)^n e^{-1} \\ &= n \cdot e\left(\frac{n}{e}\right)^n \end{aligned}$$

4: Recall that

$$\binom{n}{k} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{k(k-1)(k-1)\ldots 1} = \prod_{i=0}^{k-1} \frac{n-i}{k-i}$$

Show that

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq n^k$$

Solution: First, the upper bound is easy.

$$\binom{n}{k} = \prod_{i=0}^{k-1} \frac{n-i}{k-i} \leq \prod_{i=0}^{k-1} n - i \leq \prod_{i=0}^{k-1} n = n^k$$

Now lower bound. we will show that $\frac{n}{k} \leq \frac{n-1}{k-1}$.

$$\frac{n-1}{k-1} - \frac{n}{k} = \frac{k(n-1) - (k-1)n}{(k-1)k} = \frac{n-k}{(k-1)k} \geq 0$$

Hence

$$\binom{n}{k} = \prod_{i=0}^{k-1} \frac{n-i}{k-i} \geq \prod_{i=0}^{k-1} \frac{n}{k} = \left(\frac{n}{k}\right)^k$$

5: For $n \geq 1$ and $1 \leq k \leq n$ show that

$$\binom{n}{k} \leq \left(\frac{en}{k}\right)^k.$$

Hints: Show that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{k} \leq \left(\frac{en}{k}\right)^k.$$

using binomial theorem $(1+x)^n$ and throwing away some parts of it. At good point, use $x = \frac{k}{n}$.

Solution:

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i \geq \sum_{i=0}^{k} \binom{n}{i} x^i$$

$$\frac{1}{x^k}(1+x)^n \geq \frac{1}{x^k} \sum_{i=0}^{k} \binom{n}{i} x^i = \sum_{i=0}^{k} \binom{n}{i} x^{k-i}$$

Now if we pick $x = \frac{k}{n}$, we get $x < 1$ so $x^{k-i} \geq 1$ and we get

$$\frac{1}{x^k}(1+x)^n \geq \sum_{i=0}^{k} \binom{n}{i} \geq \binom{n}{k}$$

We are left with

$$\frac{1}{x^k}(1+x)^n = \left(\frac{n}{k}\right)^k \cdot \left(1 + \frac{k}{n}\right)^n \leq \left(\frac{n}{k}\right)^k \cdot e^{n\frac{k}{n}} = \left(\frac{en}{k}\right)^k$$

6: Show that

$$\frac{2^n}{n+1} \leq \binom{n}{\lfloor n/2 \rfloor} \leq 2^n$$

using simple arguments.

Solution: $\binom{n}{\lfloor n/2 \rfloor}$ is less than all subsets, which are 2^n.
Because there are $n+1$ and the middle is larges, it is at least the average.