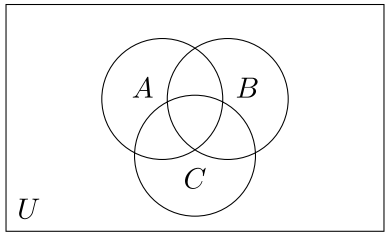
Principle of Inclusion and Exclusion - 3.7



$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

1: Find number of integers x such that $1 \leq x \leq 1000$ and x is not divisible by $5, 6, 8$.

Solution: Make sets $A_5 \ldots x$ divisible by 5, $A_6 \ldots x$ divisible by 6 and $A_8 \ldots x$ divisible by 8. We want compute

$$|U \setminus (A_5 \cup A_6 \cup A_8)| = |\overline{A_5} \cap \overline{A_6} \cap \overline{A_8}|.$$

As before we obtain

$$|\overline{A_5} \cap \overline{A_6} \cap \overline{A_8}| = |U| - \sum_{i \in \{5,6,8\}} |A_i| + \sum_{i,j \in \{5,6,8\}, i \neq j} |A_i \cap A_j| - |A_5 \cap A_6 \cap A_8|$$

Now simple counting gives

$$|A_5| = \left\lfloor \frac{1000}{5} \right\rfloor = 200 \qquad |A_6| = \left\lfloor \frac{1000}{6} \right\rfloor = 166 \qquad |A_8| = \left\lfloor \frac{1000}{8} \right\rfloor = 125$$

$$|A_5 \cap A_6| = \left\lfloor \frac{1000}{30} \right\rfloor = 33 \qquad |A_5 \cap A_8| = \left\lfloor \frac{1000}{40} \right\rfloor = 25 \qquad |A_5 \cap A_8| = \left\lfloor \frac{1000}{24} \right\rfloor = 41$$

$$|A_5 \cap A_6 \cap A_8| = \left\lfloor \frac{1000}{120} \right\rfloor = 8$$

Total gives

$$|\overline{A_5} \cap \overline{A_6} \cap \overline{A_8}| = 1000 - (200 + 166 + 125) + (33 + 25 + 41) - 8 = 600.$$

2: Let $A_1, A_2, ..., A_n$ be a collection of finite sets. Expand the formula below for $n = 3$ and compare with previous.

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{k=1}^{n} (-1)^{k-1} \left(\sum_{I \in \binom{\{1,2,\ldots,n\}}{k}} \left| \bigcap_{i \in I} A_i \right| \right)$$

Solution:

$$|A_1 \cup A_2 \cup A_3| = (|A_1| + |A_2| + |A_3|) - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3) + (|A_1 \cup A_2 \cup A_3|)$$

3: Let $A_1, A_2, ..., A_n$ be a collection of finite sets. Expand the formula below for $n = 3$ and compare with previous.

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\emptyset \neq I \subseteq [n]} (-1)^{|I|-1} \left|\bigcap_{i\in I} A_i\right|$$

Solution:

$$|A_1 \cup A_2 \cup A_3| = (|A_1|+|A_2|+|A_3|)-(|A_1\cap A_2|+|A_1\cap A_3|+|A_2\cap A_3)+(|A_1\cup A_2\cup A_3|)$$

Theorem Let S be a set and $A_1, A_2, \ldots, A_m \subseteq S$. Then

$$\left|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_m}\right| = |S| - \sum_{i\in[m]} |A_i| + \sum_{i,j\in\binom{[m]}{2}} |A_i \cap A_j| + \cdots + (-1)^m (A_1 \cap A_2 \cap \cdots A_m).$$

4: Write what the $\cdots$ stand for in the Theorem and prove it. Proof idea: Pick $x \in S$ and see how many times it is counted on each side if it it is in exactly n of the sets $A_1, \ldots, A_m$.

Solution: Left Hand Side: Elements of S in no A_i.

Right Hand Side: Let $x \in S$. Assume x in n of the sets $A_1, \ldots, A_m$, where possibly $n = 0$.

Contribution of x to the RHS:

$$n = 0 \qquad 1 - 0 + 0 - 0 + 0 = 1$$

$$n \geq 1 \qquad 1 - n + \binom{n}{2} - \binom{n}{3} + \cdots (-1)^n \binom{n}{n} + 0 \cdots + 0 = 0$$

Notice we applied binomial theorem

$$0 = (1-1)^n = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots (-1)^n \binom{n}{n}.$$

Corollary

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = \sum_{i\in[m]} |A_i| - \sum_{i,j\in\binom{[m]}{2}} |A_i \cap A_j| + \cdots + (-1)^{m+1} (A_1 \cap A_2 \cap \cdots A_m).$$

5: Explain the Corollary.

Solution: Swap the terms in the Theorem, except S to the other side.

$$\left|\bigcap_{i\in[n]} \overline{A_i}\right| = |S| + \sum_{\emptyset\neq I\in 2^{[m]}} (-1)^{|I|} \left|\bigcap_{i\in I} A_i\right|$$

$$\sum_{\emptyset\neq I\in 2^{[m]}} (-1)^{|I|+1} \left|\bigcap_{i\in I} A_i\right| = |S| - \left|\bigcap_{i\in[n]} \overline{A_i}\right| = \left|S \setminus \bigcap_{i\in[n]} \overline{A_i}\right| = \left|\overline{\bigcap_{i\in[n]} \overline{A_i}}\right| = \left|\bigcup_{i\in[n]} A_i\right|$$

6: There are 30 video game players. 15 of them play Legend of Zelda, 17 of them play Call of Duty and 20 of them play World of Warcraft. Legend of Zelda and Call of Duty is played by 8 players, Call of Duty and World of Warcraft is played by 10 players and World of Warcraft and Legend of Zelda also by 10 players. How many players play all three games?

Solution: By principle of inclusion and exclusion we have

$$30 = 15 + 17 + 20 - 8 - 10 - 10 + x$$

where x is the number of players playing all three games. Hence the result is that all three games are played by 6 players.

7: Find the number of integers between 100 and 999 inclusive that are not divisible by 4, 6, or 9.

Solution: For $i = 4, 6, 9$, let A_i denote the subset of $X = \{100, 101, \ldots, 999\}$ of integers divisible by i. Then, by the Inclusion-Exclusion Principle, the number of integers in X not divisible by any of $4, 6, 7, 10$ is

$$|X| - |A_4| - |A_6| - |A_9| + |A_4 \cap A_6| + |A_4 \cap A_9| + |A_6 \cap A_9| - |A_4 \cap A_6 \cap A_9| =$$

$$900 - (\left\lfloor \frac{999}{4} \right\rfloor - \left\lfloor \frac{99}{4} \right\rfloor) - (\left\lfloor \frac{999}{6} \right\rfloor - \left\lfloor \frac{99}{6} \right\rfloor) - (\left\lfloor \frac{999}{9} \right\rfloor - \left\lfloor \frac{99}{9} \right\rfloor) +$$

$$+(\left\lfloor \frac{999}{12} \right\rfloor - \left\lfloor \frac{99}{12} \right\rfloor) + (\left\lfloor \frac{999}{36} \right\rfloor - \left\lfloor \frac{99}{36} \right\rfloor) + (\left\lfloor \frac{999}{18} \right\rfloor - \left\lfloor \frac{99}{18} \right\rfloor) - (\left\lfloor \frac{999}{36} \right\rfloor - \left\lfloor \frac{99}{36} \right\rfloor) =$$

$$900 - 225 - 150 - 100 + 75 + 25 + 50 - 25 = 550.$$

8: Determine the number of solutions of the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 16$ in positive integers x_1, x_2, x_3, x_4 and x_5 not exceeding 7.

Solution: This number is equal to the number of nonnegatvie integer solutions of the equation $y_1 + y_2 + y_3 + y_4 + y_5 = 11$, where each y_i is at most 6. Let Y be the set of all nonnegatvie integer solutions of the equation $y_1 + y_2 + y_3 + y_4 + y_5 = 11$, and A_i be the set of solutions such that $y_i \geq 7$, $i = 1, 2, 3, 4, 5$. Since $A_i \cap A_j = \emptyset$ for all $i \neq j$, our number is

$$|Y| - |\bigcup_{i=1}^{5} A_i| = |Y| - 5|A_1| = \binom{11+4}{4} - 5\binom{4+4}{4} = 1015.$$

9: What is the number of ways to place four nonattacking rooks on the 4-by-4 boards without forbidden positions are marked by $\times$?

		$\times$	
$\times$	$\times$	$\times$	
	$\times$		$\times$

Solution: Let A_i be the set of rook placements where a rook is placed in a forbidden position in row $1 \le i \le 3$ (where the rows are numbered from top to bottom, as usual). We need to compute $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}|$ by the complementary form of the inclusion-exclusion principle.

We have

$$|A_1| = 1 \cdot 3!, |A_2| = 3 \cdot 3!, |A_3| = 2 \cdot 3!$$

$$|A_1 \cap A_2| = 1 \cdot 2 \cdot 2!, |A_1 \cap A_3| = 1 \cdot 2 \cdot 2!, |A_2 \cap A_3| = (2 \cdot 2 + 1)2!$$

and

$$|A_1 \cap A_2 \cap A_3| = (1 \cdot 1 \cdot 2 + 1 \cdot 1 \cdot 1)1!.$$

Hence the number of placements is

$$4! - 6 \cdot 3! + 18 - 3 = 24 - 36 + 18 - 3 = 3.$$

Alternatively, it is also possible to solve the problem by examining possible rook placements. Say in second row, there is only one position for placing a rook.

10: I have four exams to study for. I have 21 days to do it. I will study for any given exam during 10 days. For any given pair of exams, I will study them on the same day 5 times. For any given three exams, I will study them together on at most 3 of the days. Finally, I need to “relax” for 3 days (when I don’t study at all). How many days do I have to spend studying for all four exams together?

Solution: