Chapters 3.8 Derangements

Suppose a big crowd of people throw hats in the air. Everyone catches one hat at random. What is the probability that nobody has his/her own hat?

Formal version: Let S_n be permutations on $\{1, 2, \ldots, n\}$. Pick $\pi \in S_n$ uniformly at random. What is

$$P[\pi(i) \neq i, \forall i] =?$$

Permutation π, where $\forall i, \pi(i) \neq i$ is called a permutation **without fixed point**.
Let D_n be the number of permutations in S_n without fixed points.

1: Compute D_n using principle of inclusion and exclusion.

Solution: Let $A_i = \{\pi \in S_n, \pi(i) = i\}$.

$$\begin{aligned} D_n &= |\overline{A_1} \cap \cdots \cap \overline{A_n}| \\ &= |S_n| - \sum_i |A_i| + \sum_{i,j} |A_i \cap A_j| - \cdots (-1)^n |A_1 \cap \cdots \cap A_n| \\ &= n! - n \cdot (n-1)! + \binom{n}{2} \cdot (n-2)! - \binom{n}{3} \cdot (n-3)! + \cdots (-1)^n \binom{n}{n} \\ &= n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \cdots (-1)^n \frac{1}{n!}\right). \end{aligned}$$

2: Compute $\lim_{n\to\infty} D_n/n!$. How fast does it converge?

Solution: Recall

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots$$

Hence

$$\lim_{n\to\infty} \frac{D_n}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \cdots = e^{-1}.$$

Notice that $\left|\frac{D_n}{n!} - e^{-1}\right| < \frac{1}{n!}$. For $n = 7$, $\frac{D_n}{n!}$ and e^{-1} agree on first 3 digits. Hence the chance for a permutation without fixed point does not depend on n (very much).

3: Show that $D_n = (n-1)(D_{n-1} + D_{n-2})$. Hint: Think about $\pi(1)$ where $\pi \in S_n$ is without a fixed point.

Solution: We think what is $\pi(1)$. There are $(n-1)$ positions (the only one missing is $\pi(1) = 1$). Suppose without loss of generality that $\pi(1) = 2$. What is $\pi(2) =?$. It can be anything. If $\pi(2) = 1$, then the rest of the permutation can be filled by D_{n-2} ways. If $\pi(2) \neq 1$, then we can think that 1 *would be a fixed point for* $\pi(2)$ and the number of possible extensions is D_{n-1}. Together we get $D_n = (n-1)(D_{n-1} + D_{n-2})$.

4: Simplify the previous recurrence and prove that $D_n = nD_{n-1} + (-1)^n$. Hint slightly rewrite the previous recurrence and expand it.

Solution:

$$\begin{aligned} D_n &= (n-1)(D_{n-1} + D_{n-2}) \\ D_n - nD_{n-1} &= -(D_{n-1} - (n-1)D_{n-2}) \\ D_n - nD_{n-1} &= (-1)^2(D_{n-2} - (n-2)D_{n-3}) \\ D_n - nD_{n-1} &= (-1)^{n-2}(D_2 - (2)D_1) \\ D_n - nD_{n-1} &= (-1)^{n-2} \\ D_n &= nD_{n-1} + (-1)^n \end{aligned}$$

Recall that $D_2 = 1$ and $D_1 = 0$.

5: Use the recurrence to compute D_5.

Solution:

$$D_1 = 0 \qquad D_2 = 1 \qquad D_3 = 2 \qquad D_4 = 9 \qquad D_5 = 44$$

6: A party with 7 gentlemen. How many ways to mix theirs hats such that nobody has his own?

Solution: D_7

7: A party with 7 gentlemen. How many ways to mix theirs hats such that at least one has his own?

Solution: $7! - D_7$

8: A party with 7 gentlemen. How many ways to mix theirs hats such that at least two has their own?

Solution: $7! - D_7 - 7 \cdot D_6$ We substract if exactly one has his own.

9: Denote by $D_{n,k}$ the number of permutations in S_n with exactly k fixed points. Notice that $D_n = D_{n,0}$. Is it possible to express $D_{n,k}$ using D_m for suitable m?

Solution:

$$D_{n,k} = \binom{n}{k} D_{n-k}$$

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10 *Bonus***:** There are n canisters of gas distributed around a circular track which when all the gas is combined is exactly the amount needed for one car to make one lap of the track [the canisters are not all equally sized nor equally spaced]. Show that there is a location for a car to start with an empty tank (i.e., next to one of the canisters of gas) so that the car can make a full lap by collecting gas as it drives around the track.