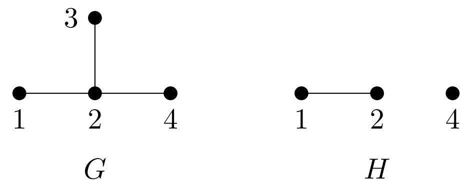
Chapters 4.2 Subgaphs

Graph H is a **subgraph** of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, notation $H \subseteq G$.



H is a **proper subgraph** if $H \subseteq G$, $H \neq G$ (and H is not a null graph).

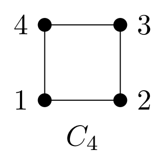
H is a **spanning subgraph** if $H \subseteq G$ and $V(H) = V(G)$

H is a **induced subgraph** if $H \subseteq G$ and $\forall u, v \in V(H), uv \in E(G) \Rightarrow uv \in E(H)$.

If $X \in V(G)$, then $G[X]$ denotes induced subgraph H of G where $V(H) = X$.

We use $-$ to denote removing edges or vertices to graph, for example $G - v$ or $G - e$. Do not use $\setminus$.

1: Count the number of subgraphs, spanning subgraphs, and induced subgraphs of C_4.



Solution: Spanning: 2^4 by deciding for each edge if it is staying or not
Induced: 2^4 by deciding for each vertex if it is staying or not
Subgraphs: By number of vertices. 4 vertices give 2^4 subgraphs. 3 vertices give 4 subgraphs each and there are 4 of them, so 4×4. On 2 vertices, the diagonal are just 2 graphs. For the 4 edges, each counts as 2 subgraphs. In total on 2 vertices, there are 10 of them. On 1 vertex, there are 4. On 0 vertices just 1. In total, $16+16+10+4+1 = 47$

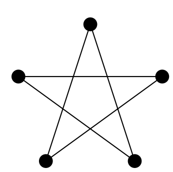
A **walk** in a graph G is a sequence $v_1, e_1, v_2, e_2, v_3, \dots, v_n$, where $v_i \in V(G)$ and $e_i \in E(G)$, where consecutive entries are incident.

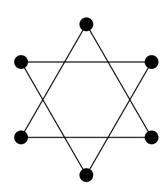
A **path** in a graph G is a walk without any repetition.

A **cycle** in a graph G is a sequence $v_1, e_1, v_2, e_2, v_3, \dots, v_n, e_n, v_1$, where $v_i \in V(G)$ and $e_i \in E(G)$, where consecutive entries are incident, and there no repetitions except v_1.

A graph G is **connected** if for every two vertices u, v, the exists a walk in G from u to v.
A **connected component** in G is a maximal subgraph of G that is connected.

2: Which of these 2 graphs is connected? Identify connected components.



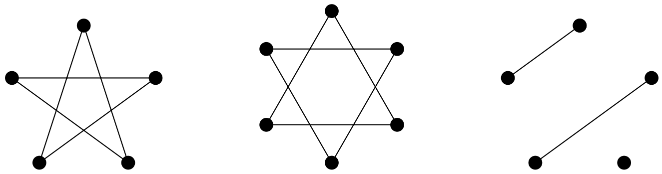


Solution: Left one is, right one is not.

The **complement** $\overline{G}$ of a graph G is graph where $V(\overline{G}) = V(G)$ and $uv \in E(G)$ iff $uv \notin E(\overline{G})$.

Complement of complete graph is **empty** graph (or **independent set**).

3: Find a complement of the following graphs.



4: Show that if G is disconnected then $\overline{G}$ is connected.

Solution: Let $V(G) = V_1 \cup V_2$, such that there are no edges between V_1 and V_2 and each is non-empty. Now verify the definition.

How to store a graph? (in a computer)

Let $G = (V, E)$ be a graph, where $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$.

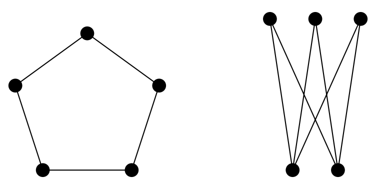
Adjacency matrix of G is $n \times n$ matrix $A = [a_{ij}]$ where

$$a_{i,j} = \begin{cases} 1 & v_i v_j \in E \\ 0 & \text{otherwise} \end{cases}$$

Incidence matrix of G is $n \times m$ matrix $B = [b_{ij}]$ where

$$b_{i,j} = \begin{cases} 1 & v_i \in e_j \\ 0 & \text{otherwise} \end{cases}$$

5: Write adjacency and incident matrices for the following graphs.



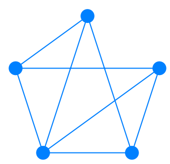
Solution: Adjacency matrices

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

6: Draw a graph G which has the following adjacency matrix:

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Solution:



7: Take the adjacency matrix A of C_4 and calculate A^2. Create a 4×4 matrix B indexed by the vertices of C_4 such that

$$B_{i,j} = \# \text{ walks of length 2 from } i \text{ to } j \text{ in the } C_4$$

Solution:

$$A^2 = B = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix}$$

Theorem Let $G = (V, E)$ be a graph with vertex set $\{v_1, \ldots, v_n\}$ and A be its adjacency matrix. Let A^k be the k-th power of A in the usual linear algebra sense and denote $A^k_{i,j}$ by $a^{(k)}_{i,j}$. Then $a^{(k)}_{i,j}$, is the number of walks in G from v_i to v_j of length exactly k.

8: Prove the theorem. Use induction on k and explore how the multiplication of matrices can correspond to extending a walk.
Hint: If I want a walk of length k, I can make it from a suitable walk of length $k-1$ and 1.

Solution:

In practice, using hash table for neighbors is perhaps best. `unordered_set` in C++.

Let $G = (V, E)$ be a connected graph. For $u, v \in V$, the **distance** of u and v is the length of the shortest path with endpoints u and v. It is denoted by $d_G(u, v)$. Notice that $d_G : V \times V \to \mathbb{R}$ is actually a *metric* on G. It satisfies the following:

1. $d_G(u, v) \geq 0$ and $d_G(u, v) = 0$ iff $u = v$
2. $d_G(u, v) = d_G(v, u)$ (symmetry)
3. $d_G(u, v) \leq d_G(u, x) + d_G(x, v)$ for any $u, v, x \in V$ (triangle inequality)

9: Show that a graph G contains K_3 (triangle) as a subgraph if and only if exists i, j such that both $A_{i,j}$ and $A^2_{i,j}$ are non-zero. Recall that A is the adjacency matrix of G.

Remark. Finding a triangle in a graph can be obviously done in $O(n^3)$. This provides the fastest known algorithm to find one, it is based on the matrix multiplication, with can be done in $O(n^{2.373})$.

Solution:

10: Show that a graph G is bipartite if and only if it does not contain an odd cycle as a subgraph.

Solution:

11: Let G be a connected graph that has neither C_3 nor P_4 as an induced subgraph. Prove that G is a complete bipartite graph.

Solution: