# Chapters 4.3 Graph Score (degree sequence)

Let $G = (V, E)$ be a graph. For a vertex $v \in V$, denote the set of neighbors of $v$ in $V$ by $N(v)$. The **degree** of $v$, denote by $\deg(v)$ or $d(v)$, is $|N(V)|$. If the graph $G$ is not clear from the context, we use $\deg_G(v)$ or $d_G(v)$.

Let $v_1, v_2, \ldots, v_n$ be the vertices of $V$ in some order. The **degree sequence** or **score** of $G$ is

$$(\deg(v_1), \deg(v_2), \ldots, \deg(v_n)).$$

**1:** Write the degree sequences for the following graphs.

**Solution:** For example $(1, 1, 1, 3)$ or $(1, 3, 1, 1)$. Notice that the degree sequence is not unique. It depends on the order of vertices. On the right, it is just $(2, 2, 2, 2, 2)$

**2:** Find two non-isomorphic graphs with degree sequence $(2, 2, 2, 2, 2, 2)$.

**Solution:**

Vertex $v$ is **isolated** if $d(v) = 0$.

Vertex $v$ is **leaf** if $d(v) = 1$.

The **minimum degree** of $G$ is $\delta(G) = \min_{v \in V(G)} d(v)$.

The **maximum degree** of G is $\Delta(G) = \max_{v \in V(G)} d(v)$.

**3** ***Handshaking Lemma*:** Show that if a graph $G$ has $m$ edges when

$$\sum_{v \in V(G)} deg(v) = 2m$$

**Solution:** Every edge contributes 1 to degree of its endpoints.

**4:** Show that every graph has an even number of vertices of odd degree.

**Solution:** By the Handshaking Lemma, the sum of degrees is even. It means that the sum of degrees of vertices of odd degree is also even. Hence there must be even number of them.

Graph $G$ is $r$**-regular** if $r = \delta(G) = \Delta(G)$.

3-regular graphs are called **cubic**.

**5:** Find two cubic graphs.

**Solution:** For example $K_4$. Or any 2 cycles and edges between them.

**6:** Let $r$ and $n$ be integers with $0 \leq r \leq n-1$. Show that there exists an $r$-regular graph of order $n$ if and only if at least one of $r$ and $n$ is even. (Harary graphs) *Hint: Try for small $r$.*

**Solution:** If both are odd, clearly the graphs cannot exist. If $r$ is even, we can try to make cycles that contain all vertices (i.e. Hamiltonian cycles). Particular example: Vertices $v_0, v_1, \ldots, v_n$. Add edges $v_i, v_{(i+j) \bmod n}$ for all $i$ and $1 \leq j \leq \lfloor r/2 \rfloor$. If $r$ is odd and $n$ even, do a same construction and add edges $v_i, v_{i+n/2}$.

A finite sequence of nonnegative integers is **graphical** if it is a degree sequence of some graph.

**Theorem Havel-Hakimi** A non-decreasing sequence $s : 0 \leq d_1 \leq d_2 \leq \cdots \leq d_n$ of non-negative integers is graphical if and only if the sequence $s_1 : d'_1, d'_2, \ldots, d'_{n-1}$, where

$$d'_i = \begin{cases} d_i & \text{if } i < n - d_n \\ d_i - 1 & \text{if } i \geq n - d_n \end{cases}$$

is graphical.

**7:** Decide if there exists a graph with degree sequence $(1, 1, 1, 2, 2, 3, 4, 5, 5)$.

**Solution:** We apply the theorem a couple times

$$(1,1,1,2,2,3,4,5,5) \to (1,1,1,1,1,2,3,4) \to (1,1,1,0,0,1,2) \equiv (0,0,1,1,1,1,2) \to$$
$$\to (0,0,1,1,0,0) \equiv (0,0,0,0,1,1) \to (0,0,0,0)$$

The last is an empty graph on 4 vertices. So it is easily a graphical sequence.

**8:** Is $2, 2, 2, 2, 2, 3, 3, 5, 5$ graphical? Justify your answer.

**Solution:** We apply the theorem a couple times

$$(1,1,2,2,2,5,5,6,6) \to (1,1,1,1,1,4,4,5) \equiv (0,0,0,1,1,3,3) \to$$
$$\to (0,0,0,0,0,2) \equiv (0,0,0,-1,-1)$$

Well, it is hard to have a graph with negative degrees.

**9:** Proof one of the implications of the theorem by showing that if $s_1$ is graphical, then $s$ is also graphical.

**Solution:** Take a graph $G_1$ whose degree sequence is $s_1$. Assume that vertex $v_i$ has degree $d_i$. Add a new vertex $v_n$ to the graph and make it adjacent to vertices $v_{n-1}, \ldots, v_{n-d_n}$. The new graph has a degree sequence $s$.

**10:** Proof the second implication showing that if $s$ is graphical, then $s_1$ is also graphical. Hint: Try to find a graph, where the previous argument could be reversed.

**Solution:** Take a graph $G$ whose degree sequence is $s$. Assume that vertex $v_i$ has degree $d_i$. If $v_n$ is adjacent to vertices $v_{n-1}, \ldots, v_{n-d_n}$, we can remove it and get a graph with degree sequence $s_1$.

Take a graph, where $v_n$ with degree sequence $s$, where $v_n$ has as many neighbors among $v_{n-1}, \ldots, v_{n-d_n}$ as possible. Since not all neighbors are there, It means that there exists $i < n - d_n \leq j$ such that $v_n v_i$ is an edge and $v_n v_j$ is not an edge. Since $d_i \leq d_j$, exists a vertex $z$ adjacent to $v_j$ not adjacent to $v_i$. Swap edges $v_j z$ and $v_n v_i$.