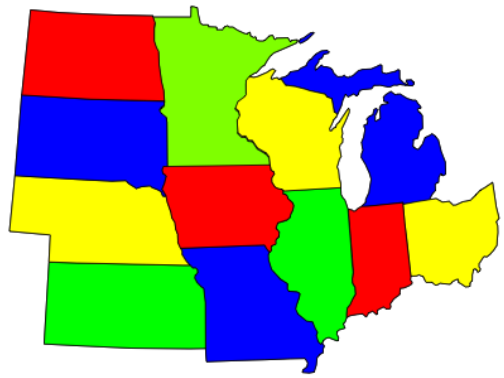
Chapter 6.4 Graph coloring

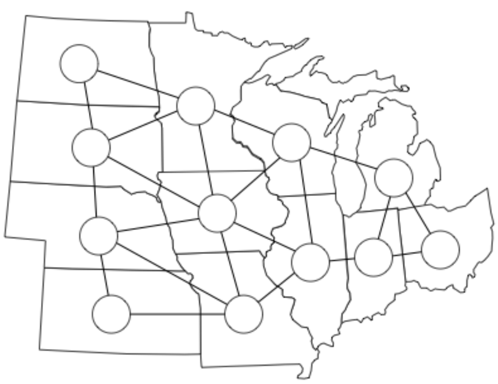
Problem: Color regions of the plane such that regions sharing border get different colors. Show that 4 colors is enough (if regions connected) for any set of regions.
(Restating: Is it true that every planar graph is 4-colorable? Answer is yes.)

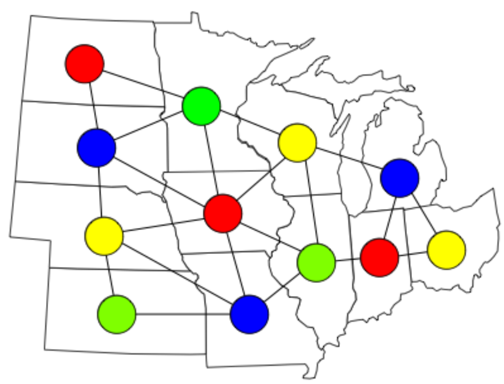
1: Color the states in midwest by 4 colors with neighboring ones have distinct colors. Can you do 3 colors?



The problem can be turned into a graph problem by having a vertex for every region.

2: Translate your coloring to a coloring of the graph below.





Let G be a graph and C be a set of colors. A **coloring** is a mapping $c : V(G) \to C$ such that $c(u) \neq c(v)$ for all $uv \in E(G)$. Sometimes called **proper coloring**.

A graph G is k**-colorable** is there exists a (proper) coloring of G using k colors.

Chromatic number of G, denoted by $\chi(G)$ is the minimum k such that G is k-colorable.

3: Decide what is the chromatic number of C_k. (try $3 \leq k \leq 7$)

Solution: $\chi(C_4) = \chi(C_6) = 2$ and $\chi(C_3) = \chi(C_5) = \chi(C_7) = 3$.
For all even cycles, $\chi(C_{2k}) = 2$ and for odd cycles $\chi(C_{2k+1}) = 3$.

A set of vertices $X \subset V(G)$ are **independent** in a graph G if $G[X]$ has no edges.

Let c be a (proper) coloring of G. If V_{red} is the set of vertices colored red then V_{red} is an independent set.

Coloring G by k colors is a decomposition of $V(G)$ into in k independent sets. $V(G) = V_1 \cup V_2 \cup \cdots \cup V_k$.

4: Show that $\chi(G) = 2$ iff G is bipartite (and has at least one edge).

Solution: If G is 2-colorable, then each of the two color classes is an independent set and it creates the desired bipartition. If G is bipartite, color one class with color one and the other with color two. That creates the desired coloring.
Notice that G is bipartite iff it does not contain an odd cycle as a subgraph. Hence we get a characterization that G is 2-colorable iff G does not contain an odd cycle as a subgraph. No nice characterization is known for more than 2 colors.

5: What is $\chi(K_n)$?

Solution: $\chi(K_n) = n$ since everyone is connected with everyone. This gives an easy lower bound.

A **clique** in a graph G is a subgraph that is isomorphic to a complete graph.
The **clique number**, $\omega(G)$, is the order of the largest clique in G.

Recall $\Delta(G)$ is the maximum degree of a vertex in G.

6: Show that for every graph G holds $\omega(G) \leq \chi(G) \leq \Delta(G) + 1$.

Solution: Vertices in a clique has to be colored by different colors. Hence $\omega(G) \leq \chi(G)$. For the other bound, color the vertices one by one. Everytime you are about to color a vertex, it has at most $\Delta(G)$ neighbors that are already colored. So it needs to avoid at most $\Delta(G)$ colors, but there are $\Delta(G) + 1$ colors available.

Theorem [Brook's]. Let G be a connected graph, that is not a complete graph or an odd cycle. Then $\chi(G) \leq \Delta(G)$.

Notice that Brook's theorem tells us that $\chi(G) \leq \Delta(G) + 1$ holds with equality iff G is a complete graph or an odd cycle.

Let's prove Brook's theorem. Let G be a connected graph that is not complete or an odd cycle. Also assume that we have proved the theorem for all smaller graphs (we are proving by induction on the number of vertices). Let $\Delta(G) = \Delta$.

7: Show that the Brook's theorem holds if $\Delta = 2$

Solution: If $\Delta = 2$, then G is a cycle or a path and checking the Brook's claim is straightforward.

So we assume $\Delta \geq 3$.

8: Show that G is 2-connected. (use induction)

Solution: If G not 2-connected, one can color all blocks of G separately with Δ colors and combine them by permuting colors. Recall that block is a maximal 2-connected subgraph.

9: Prove the case where G has a vertex v of degree less than Δ (greedy coloring with v last)
Hint: Use a spanning tree to make an ordering where everyone bu v has a neighbor that comes later.

Solution: Take any spanning tree T of G. Orient all edges of T towards v. Now order the vertices of G such that if xy is an edge of T oriented in this direction, then x is in the ordering before y. Hence v is last in the ordering and every vertex has at least one neighbor behind in the ordering. Now try to color vertices according to this ordering. Every vertex has to avoid color of at most $\Delta - 1$ other vertices, that are in the ordering before. So Δ colors is enough.

Now we assume G is Δ-regular. We still want to use greedy coloring, but guarantee that the last vertex has 2 neighbors with the same color.

10: Assume that there is a vertex v such that $G - v$ is 2-connected. Prove Brook's Theorem.
Hint: Start as: Take v and any vertex y in distance 2 from it (why it exists?). They have a common neighbor z. Since $G - v$ is 2-connected, $G - v - y$ is connected. Hence there is an orientation of edges of $G - v - y$ that all go towards z.

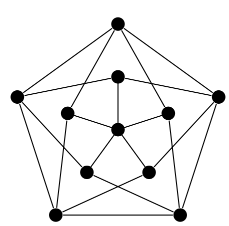
Solution: So we there is an ordering of $G - v - y$ where z is last and all vertices but z have a neighbor behind in the ordering. Now we create the final ordering by putting the vertices v and y first. Notice that in a coloring, v and y get the same color since they come first and they are not adjacent. So z has Δ neighbors that are already colored when coloring z, but two of them have the same color. Hence Δ colors is enough.

11: Assume that there is a vertex v such that $G - v$ is not 2-connected. Prove Brook's Theorem. Notice that $G - v$ is still connected. Consider block decomposition of $G - v$ and see where are neighbors of v.

Solution: Since G is 2-connected, v has to have neighbors in the *end*-blocks. Take neighbors x and y of v in two different end-blocks. Notice that $G - x - y$ is still connected. Create an ordering where x and y are first and v is last.

Question: Do you need a large clique in a graph G for a large chromatic number?

12: What is the chromatic number of the Grötzsch's graph? Notice it is triangle-free.



Solution: It is 4. There is a coloring with 4 colors. For showing that 3 is not enough, try to do a 3-coloring. First color the outer 5-cycle. Then vertices in the inner 5-cycle must still contain all three colors. This kills the last color for the middle vertex.

Theorem (Erdős) For every k, ℓ there exists a graph of girth ℓ and chromatic number k.

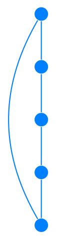
The proof is probabilistic and we skip it.

Mycielski construction is a construction to create a triangle free graph of arbitrary chromatic number.

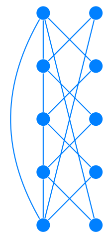
Start with a graph G, duplicate every vertex and connect new vertex to the duplicates.

13: Apply the Mycielski operation on C_5.

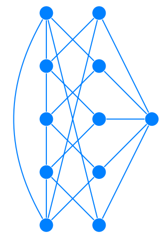
Solution:



C_5



C_5 duplicated vertrices



The whole thing

Notice that this graph is isomorphic to the Grötzsch's graph.

14: Let G be a graph with $\chi(G) = k$ and c be a k-coloring of G. Show that for every color z, there exists a vertex $v \in V(G)$ such that all remaining $k-1$ are on neighbors of v.
Hint: Show that if the conclusion is not true, then $\chi(G) < k$.

Solution: Suppose for contradiction that there exists a color z such that for every vertex v of color z exists a color $a_v \neq z$ that is not on any of the neighbors of v. The v can be recolored from z to a_v. If we do it for all vertices colored by z, there will be none left. Notice that vertices colored by z form an independent set so this operation will not create new conflicts.

15: Show that the Mycielski construction is increasing the chromatic number.

Solution: Suppose the original graph G that went into the construction has $\chi(G) = k$. If we k-color it, for every color there is a vertex c such that its neighbors have all remaining $k-1$ colors. Hence the duplicate of v must have the color of v. The last vertex added will have neighbors of all k colors. Hence the graph that came from the construction is not k-colorable and at least one extra color is needed.

Theorem Every planar graph is 4-colorable.
A serious theorem. No simple proof known. Computer assisted.

16: Show that every planar graph is 6-colorable. Recall that $\delta(G) \leq 5$.

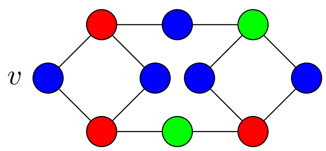
Solution: We use proof by induction. Let G be a planar graph on n vertices and graphs on $n-1$ vertices are 6-colorable. Let v be a vertex of G of degree 5. By induction, $G-v$ is 6-colorable by coloring c. Since v has only 5 neighbors, there is at least one if the 6 colors not used on neighbors of v and c can be extended to a 6-coloring of G.

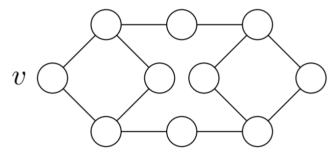
Theorem Every planar graph is 5-colorable.

We do the proof by induction on the number of vertices. We use Kempe chains.

Let c be a coloring of G. A **Kempe chain** in colors 1 and 2 is a maximal connected subgraph of G where all vertices are colored 1 and 2.

17: In the following graph, take a Kempe chain in colors red and blue that contains the vertex v and swap *red* and *blue* colors on it. Draw the result on the right.





18: Let $K \subseteq V(G)$ be a Kempe chain in G for a coloring c. Let c' be obtained from c by swapping colors 1 and 2 on K but nowhere else.

$$c'(v) = \begin{cases} 1 & \text{if } v \in K \text{ and } c(v) = 2 \\ 2 & \text{if } v \in K \text{ and } c(v) = 1 \\ c(v) & \text{otherwise} \end{cases}$$

Show that c' is a proper coloring.

Solution: Suppose that c' is not a proper coloring. Assume that there are two vertices u and v such that $c'(u) = c'(v) = a$. Since it only changes colors are 1 and 2, it must hold that $a \in \{1, 2\}$. If both $u, v \in K$, then they should have different colors. Same if both $u, v \notin K$. Hence we conclude $u \in K$ and $v \notin K$. This is a contradiction with the maximality of K.

Proof of the 5-color theorem. Let G be a plane graph on n vertices. Assume that all planar graphs on at most $n-1$ vertices are 5-colorable.

19: Show that G is 5-colorable if it contains a vertex v of degree at most four.

Solution: If G contains a vertex v of degree at most 4, we consider a 5-coloring of $G-v$ and extend it to a 5-coloring of G since there are at most 4 colors used on the neighbors of v.

Let v be a vertex of degree 5 in G. By induction, there is a 5-coloring c of $G - v$. If the neighbors of v use at most 4 colors in c, the coloring c can be extended to v. Hence assume that the neighbors of v are colored by $1, 2, 3, 4, 5$ (in clockwise order in the drawing on G).

20: Use Kempe chain in colors 1 and 3 and another one in colors 2 and 4 to show that there exists a coloring c' of $G - v$ such that the neighbors of G have at most 4 colors.

Solution: Let v_i be a neighbor of v colored by color i. First we try to take the Kempe chain $K_{1,3}$ in colors 1 and 3 that contains v_1. If $K_{1,3}$ does not contain v_3, then switching colors on $K_{1,3}$ creates a coloring that can be extended to v by coloring v by 1. So assume that $v_3 \in K_{1,3}$. So we take a Kempe chain $K_{2,4}$ in colors 2 and 4 that contains v_2. Observe that by planarity, $v_4 \notin K_{2,4}$. Hence we can flip colors on $K_{2,4}$ and color v by color 2. This finishes the proof of 5-color theorem.