# Genuine Quotes From Last Year

- I thought, because I took Calculus in high school, that I could do all the things an ‘F student’ does and still get an A.

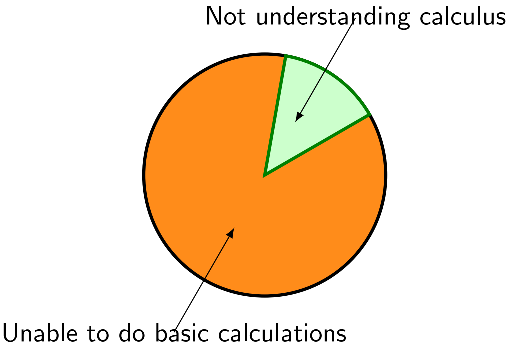
- Three hours a week was not enough time for this instructor to teach and explain all the topics and I've taken calc I before.

# Understanding is MORE Important Than Answer

1) Who took calculus at high school? 2) Who could say what is derivative of $x^2$? 3) Who could say what is derivative?

- Understanding WHAT are you doing is important
- Answer alone is useless
- No need for calculators (solution steps matter)

# If you are not ready, take MATH-143 Precalculus



# Chapter 2.2. - Limit of a Function

# What a Slope Should Be?

Recall: Slope of secant line is $\frac{f(b)-f(a)}{b-a}$.

Slope of tangent line would be slope of secant line with $a = b$.

$\frac{f(b)-f(a)}{b-a} = \frac{f(a)-f(a)}{a-a} = \frac{0}{0}$

*Division by zero is bad!*

$\frac{a}{b} = c$ whenever $a = bc$. Then $\frac{a}{0} = c$ gives $a = c0$. $a = 0$ and $b$ can be anything!

Solution: Study $\frac{f(a+h)-f(a)}{h}$ where $h$ is going to zero and find what it *should* be.

Example: Find tangent of $f(x) = \frac{1}{2}x^2 + 1$ at $a = 2$.

| $h$ | 1 | 0.5 | 0.1 | 0.01 | 0.001 |
|---|---|---|---|---|---|
| $\frac{f(2+h)-f(2)}{h}$ | 2.5 | 2.25 | 2.05 | 2.005 | 2.0005 |

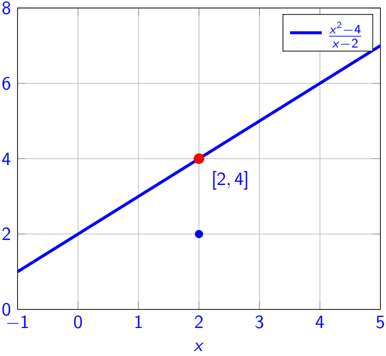
What *should* be $\frac{f(2+h)-f(2)}{h}$ for $h = 0$?

# Limit of $f(x)$ at $x_0$

If $f(x)$ approaches $L$ as $x$ approaches $x_0$ we write

$$\boxed{\lim_{x \to x_0} f(x) = L}$$

approximate $f(x_0)$ by $f(x)$ around $x_0$
$f(x_0)$ may be undefined
maybe $f(x_0) \neq \lim_{x \to x_0} f(x)$
$x$ is arbitrarily close to $x_0$ *but not* $x_0$

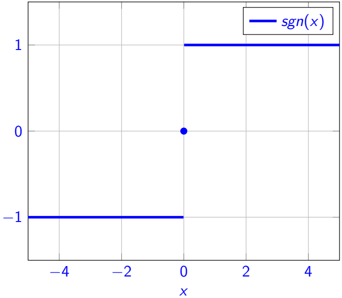


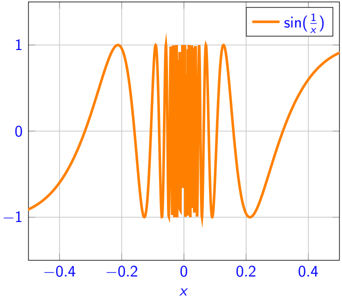

Example: Let $f(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{if } x \neq 2 \\ 2 & \text{if } x = 2 \end{cases}$.

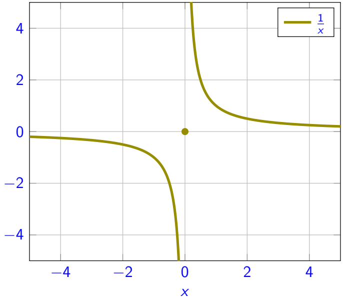
$\lim_{x \to 3} f(x) = 5$

$\lim_{x \to 2} f(x) = 4$

# Limit May Not Exist



# Basic Properties Of Limits

Let $f$ and $g$ be functions and $c \in \mathbb{R}$ a constant.

- $f(x) = c$. $c$ is always near to $c$

$$\lim_{x \to a} c = c \qquad \lim_{x \to 3} 4 = 4$$

- $f(x) = x$. If $x$ is close to $a$ then $f(x)$ is close to $a$.

$$\lim_{x \to a} x = a \qquad \lim_{x \to \pi} x = \pi$$

- Multiplying by scalar

$$\lim_{x \to a} c \cdot f(x) = c \cdot \lim_{x \to a} f(x) \qquad \lim_{x \to 7} 2x = 2 \lim_{x \to 7} x = 2 \cdot 7 = 14$$

- Addition

$$\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

$$\lim_{x \to 3} (2x + 4) = \lim_{x \to 3} 2x + \lim_{x \to 3} 4 = 2 \lim_{x \to 3} x + 4 = 2 \cdot 3 + 4 = 10$$

*If right hand side makes sense!*

Limits are linear.

# More Arithmetics With Limits

- Multiplication

$$\lim_{x\to a}(f(x)\cdot g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$

$$\lim_{x\to 2}(x\cdot 2x) = \left(\lim_{x\to 2} x\right)\cdot\left(\lim_{x\to 2} 2x\right) = 2\cdot 4 = 8$$

- Division

$$\lim_{x\to a}(f(x)/g(x)) = \lim_{x\to a} f(x) / \lim_{x\to a} g(x)$$

$$\lim_{x\to 2}\frac{2x+3}{x-9} = \frac{\lim_{x\to 2} 2x+3}{\lim_{x\to 2} x-9} = \frac{7}{-7} = -1$$

- Power

$$\lim_{x\to a} f(x)^r = \left(\lim_{x\to a} f(x)\right)^r \qquad \lim_{x\to 4}(x+1)^3 = \left(\lim_{x\to 4} x+1\right)^3 = 5^3 = 125$$

*If right hand side makes sense!*

# Everybody Loves Polynomials

Example:
$\lim_{x\to 1} x^3 - 3x + 1 = \lim_{x\to 1} x^3 + \lim_{x\to 1} -3x + \lim_{x\to 1} 1 = 1^3 - 3 + 1 = -1$

Polynomial: $f(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 = \sum_{j=0}^{n} c_j x^j$

$$\begin{aligned}
\lim_{x\to a} f(x) &= \lim_{x\to a} (c_n x^n + \cdots + c_1 x + c_0) \\
&= \lim_{x\to a} (c_n x^n) + \cdots + \lim_{x\to a} \left(c_0 x^0\right) \\
&= c_n \lim_{x\to a} (x^n) + \cdots + c_0 \lim_{x\to a} \left(x^0\right) \\
&= c_n \left(\lim_{x\to a} x\right)^n + \cdots + c_0 \left(\lim_{x\to a} x\right)^0 \\
&= c_n a^n + \cdots c_1 a^1 + c_0 a^0 \\
&= f(a)
\end{aligned}$$

Example: $\lim_{x\to 3} \frac{x^2-2x+1}{x-1} =$
$= \frac{9-6+1}{4} = 1$

Example: $\lim_{x\to 1} \frac{x^2-2x+1}{x-1} = \frac{0}{0}$
does not make sense, too bad.

Polynomials are great! $\lim_{x\to a} f(x) = f(a)$. Easy to find limits!

# Tricks For Evaluating $\frac{0}{0}$

Example: $\lim_{x\to 3} \frac{4(x-3)}{(x-3)} = 4$

$\boxed{\text{Expanding/Factoring polynomials}}$

Example:

$$\lim_{x\to\frac{1}{2}} \frac{2x^2+5x-3}{10x-5} == \lim_{x\to\frac{1}{2}} \frac{2x^2+5x-3}{10x-5}$$

$$= \lim_{x\to\frac{1}{2}} \frac{(2x-1)(x+3)}{5(2x-1)} = \lim_{x\to\frac{1}{2}} \frac{x+3}{5} = \frac{3.5}{5}$$

We can do division of polynomial by polynomial:

$$2x^2+5x-3 : 2x-1 = x+3$$

# Tricks For Evaluating $\frac{0}{0}$

Recall: $(a+b)(a-b) = a^2 - b^2$

Multiplying by 1 $(= \frac{c}{c})$

Example:

$$\begin{aligned}
\lim_{x\to 3} \frac{\sqrt{x+1}-2}{\sqrt{x+6}-3} &= \lim_{x\to 3} \frac{\sqrt{x+1}-2}{\sqrt{x+6}-3} \cdot \frac{\sqrt{x+6}+3}{\sqrt{x+6}+3} \\
&= \lim_{x\to 3} \frac{(\sqrt{x+1}-2)(\sqrt{x+6}+3)}{x-3} \\
&= \lim_{x\to 3} \frac{(\sqrt{x+1}-2)(\sqrt{x+6}+3)}{x-3} \cdot \frac{\sqrt{x+1}+2}{\sqrt{x+1}+2} \\
&= \lim_{x\to 3} \frac{(x-3)(\sqrt{x+6}+3)}{(x-3)(\sqrt{x+1}+2)} \\
&= \lim_{x\to 3} \frac{\sqrt{x+6}+3}{\sqrt{x+1}+2} = \frac{\sqrt{3+6}+3}{\sqrt{3+1}+2} = \frac{6}{4} = \frac{3}{2}
\end{aligned}$$

# Tricks For Evaluating $\frac{0}{0}$

Recall: $\sin(x)^2 + \cos(x)^2 = 1$

$$\boxed{\text{Use Identities (trigonometry)}}$$

Example:

$$\lim_{x \to 0} \frac{\sin^2 x}{1 - \cos x} = \lim_{x \to 0} \frac{1 - \cos^2 x}{1 - \cos x} = \lim_{x \to 0} \frac{(1 - \cos x) \cdot (1 + \cos x)}{1 - \cos x}$$
$$= \lim_{x \to 0} 1 + \cos x = 2$$

# Tricks For Evaluating $\frac{0}{0}$

Clever substitution

Example: $\lim_{z\to 0} \frac{\sqrt[3]{1+z}-1}{z} =$

Alright, we need to get crafty with this one. Let

$$x = \sqrt[3]{1+z}.$$

As $z \to 0$, then $x \to 1$. Solving for $z$ yields

$$z = x^3 - 1.$$

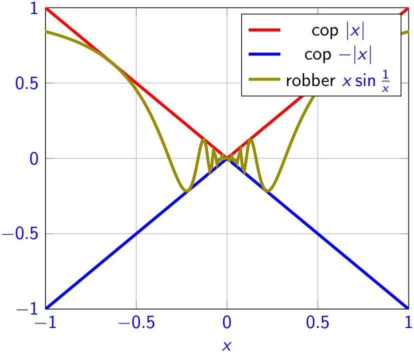
Consequently, we can rewrite our limit in terms of this new variable:

$$\lim_{z\to 0} \frac{\sqrt[3]{1+z}-1}{z} = \lim_{x\to 1} \frac{x-1}{x^3-1} = \lim_{x\to 1} \frac{x-1}{(x-1)(x^2+x+1)} = \lim_{x\to 1} \frac{1}{x^2+x+1} = \frac{1}{3}.$$

# Squeeze Theorem - About Two Cops

## Theorem (Sandwich, Squeeze, *About 2 cops*)

*If* $g(x) \leq f(x) \leq h(x)$ *near* $c$ *and* $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$, *then* $\lim_{x\to c} f(x) = L$.




Example: Compute $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right)$

Notice $-1 \leq \sin(x) \leq 1$ is true for any $x$, we have that

$$-|x| \leq x \sin\left(\frac{1}{x}\right) \leq |x|$$

As

$$\lim_{x\to 0} -|x| = \lim_{x\to 0} |x| = 0,$$

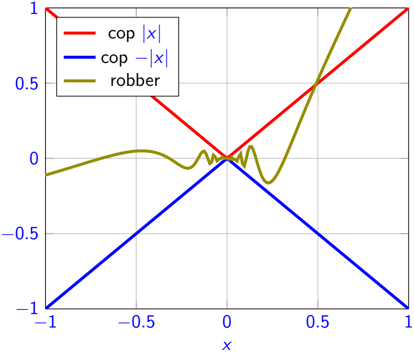
the 2 Cops Theorem tells us that

$$\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0.$$

# Squeeze Theorem - About Two Cops

## Theorem (Sandwich, Squeeze, *About 2 cops*)

*If* $g(x) \le f(x) \le h(x)$ *near* $c$ *and* $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$, *then* $\lim_{x\to c} f(x) = L$.




Example: Compute

$$\lim_{x\to 0} (e^x - 1)x \sin\left(\frac{1}{x}\right)$$

$-1 \le \sin(x) \le 1$ is true $\forall x$
$-0.5 \le e^x - 1 \le 1$ is true for $x \le \ln 2$

$$-|x| \le (e^x - 1)x \sin\left(\frac{1}{x}\right) \le |x|$$

As

$$\lim_{x\to 0} -|x| = \lim_{x\to 0} |x| = 0,$$

the 2 Cops Theorem implies $\lim_{x\to 0}(e^x - 1)x \sin\left(\frac{1}{x}\right) = 0$.

# Chapter 2.2 Recap

- Limit of $f(x)$ at $a$, what $f(x)$ should be?
- Limit may be undefined.
- Limits are linear.
- It is also easy to multiply, divide, take power.
- Limits of polynomials are easy.
- Tips and Tricks for $\frac{0}{0}$.
- 2 cops Theorem