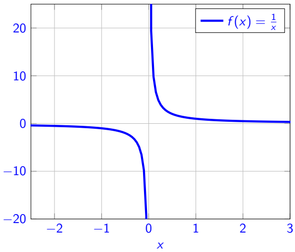
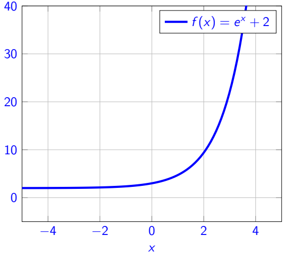
Chapter 2.6 - Limits Involving Infinity; Asymptotes of Graphs

Infinity, $-\infty$, $+\infty$

$x \to \infty$ means x is getting bigger and bigger. ∞ is not some really large number.

Example:





- $\lim_{x \to \infty} \frac{1}{x} = 0$
- $\lim_{x \to -\infty} \frac{1}{x} = 0$
- $\lim_{x \to 0^+} \frac{1}{x} = \infty$
- $\lim_{x \to 0^-} \frac{1}{x} = -\infty$
- $\lim_{x \to \infty} e^x + 2 = \infty$
- $\lim_{x \to -\infty} e^x + 2 = 2$

Horizontal Asymptotes

Situation where $f(x)$ approaches a particular value as $x \to \infty$ and/or $x \to -\infty$.

$$\boxed{\lim_{x\to\infty} f(x) = \kappa \qquad \lim_{x\to-\infty} f(x) = \kappa}$$

$f(x)$ is similar to a horizontal line as $x \to \infty$

Always one sided limits.
Fun when $\frac{\infty}{\infty}$

Approach: For $\frac{\infty}{\infty}$ look for a way to divide out by the fastest growing thing (pull on the reins so the parts don't head off to ∞).

$$\lim_{x\to\infty} \frac{8e^x+3}{1+2e^x} = \lim_{x\to\infty} \frac{e^x\left(8+\frac{3}{e^x}\right)}{e^x\left(2+\frac{1}{e^x}\right)}$$

$$\lim_{x\to\infty} \frac{8+\frac{3}{e^x}}{2+\frac{1}{e^x}} = \frac{8}{2}$$

Horizontal Asymptotes

$$\lim_{x\to\infty} f(x) = \kappa \qquad \lim_{x\to-\infty} f(x) = \kappa$$

If $\lim_{x\to\infty} f(x) = \infty - \infty$ try to make a ratio (conjugation).

$$\lim_{x\to-\infty} \left(\sqrt{x^2+4x+7} - x\right) = \infty$$

$$\lim_{x\to\infty} \left(\sqrt{x^2+4x+7} - x\right) = \lim_{x\to\infty} \left(\sqrt{x^2+4x+7} - x\right) \frac{\left(\sqrt{x^2+4x+7} + x\right)}{\left(\sqrt{x^2+4x+7} + x\right)}$$

$$= \lim_{x\to\infty} \frac{x^2+4x+7-x^2}{\sqrt{x^2+4x+7}+x} = \lim_{x\to\infty} \frac{x^2+4x+7-x^2}{\sqrt{x^2+4x+7}+x} =$$

$$\lim_{x\to\infty} \frac{4x+7}{\sqrt{x^2(1+\frac{4}{x}+\frac{7}{x^2})}+x} = \lim_{x\to\infty} \frac{x(4+\frac{7}{x})}{x\sqrt{1+\frac{4}{x}+\frac{7}{x^2}}+x} =$$

$$\lim_{x\to\infty} \frac{4+\frac{7}{x}}{\sqrt{1+\frac{4}{x}+\frac{7}{x^2}}+1} = \frac{4}{2} = 2$$

Vertical Asymptotes

Situation where $f(x)$ approaches $\pm\infty$ as $x \to a$ for some $a \in \mathbb{R}$.

$$\boxed{\lim_{x\to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = \pm\infty}$$

$f(x)$ would have a tangent vertical line as $x \to a$.

Usually one sided limits.

Approach: Vertical asymptote of $f(x)$ is blowup-up (or down) near a. If $\lim_{x\to a^\pm} f(x)$ goes to $\frac{c}{0}$ for some $c \in \mathbb{R}$, it is a sign of vertical asymptote.
Look what happens near a.

$$\lim_{x\to 1} \frac{3x^2 - 7x + 5}{x^3 - 2x^2 + x} = \frac{3-7+5}{1-2+1} = \frac{1}{0}$$

$$\frac{\lim_{x\to 1} 3x^2 - 7x + 5}{\lim_{x\to 1} x^3 - 2x^2 + x} =$$

$$\frac{1}{\lim_{x\to 1} x(x^2 - 2x + 1)} =$$

$$\frac{1}{\lim_{x\to 1} x \lim_{x\to 1}(x-1)^2} =$$

$$\frac{1}{\lim_{x\to 1}(x-1)^2} = \frac{1}{\lim_{x\to 0} x^2} = \infty$$

Vertical Asymptotes

Situation where $f(x)$ approaches $\pm\infty$ as $x \to a$ for some $a \in \mathbb{R}$.

$$\boxed{\lim_{x\to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = \pm\infty}$$

Example: Find $\displaystyle\lim_{t\to 0} \frac{1}{t} \sin\left(\frac{1}{t}\right)$

One could think that $-1 \leq sin(1/t) \leq 1$ makes it a fixed number. And then $\lim_{t\to 0} \frac{1}{t}$ is blowing up. However, the $sin(1/t)$ is oscillating its sign so the limit does not exists. It just jumps up and down. Draw a nice figure.

Example

Find all asymptotes for $f(x)=\frac{x^2+4x+3}{x^2-2|x|+1}$ $\quad f(x)=\begin{cases}\frac{x^2+4x+3}{x^2-2x+1} & \text{if } x\geq 0\\ \frac{x^2+4x+3}{x^2+2x+1} & \text{if } x\leq 0\end{cases}$

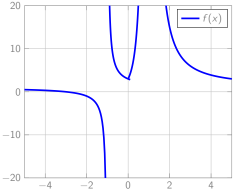
$$f(x)=\begin{cases}\frac{(x+1)(x+3)}{(x-1)^2} & \text{if } x\geq 0\\ \frac{(x+1)(x+3)}{(x+1)^2}=\frac{x+3}{x+1} & \text{if } x\leq 0\end{cases}$$

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{x^2+4x+3}{x^2-2x+1}=\lim_{x\to\infty}\frac{x^2(1+4/x+3/x^2)}{x^2(1-2/x+1/x^2)}=1$$

$$\lim_{x\to-\infty} f(x)=\lim_{x\to\infty}\frac{x+3}{x+1}=\lim_{x\to\infty}\frac{x(1+3/x)}{x(1+1/x)}=1$$

$$\lim_{x\to 1} f(x)=\lim_{x\to 1}\frac{(x+1)(x+3)}{(x-1)^2}\to\frac{8}{0^+}\text{ From both sides positive}$$

$$\lim_{x\to -1} f(x)=\lim_{x\to -1}\frac{x+3}{x+1}\to\frac{2}{0}\text{ Note }\lim_{x\to -1^-} f(x)=-\infty\text{ and }\lim_{x\to -1^+} f(x)=\infty$$



Chapter 2.6 Recap

- ∞ means something is growing (is unbounded)
- Horizontal asymptote if $\lim_{x\to\infty} f(x) = \kappa$ $\qquad \lim_{x\to-\infty} f(x) = \kappa$
- Vertical asymptote if $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$
- Careful about sign when evaluating $\frac{\textit{something}}{0}$
- $\frac{0}{0}$, $\frac{\infty}{\infty}$, and $\infty - \infty$ need more work