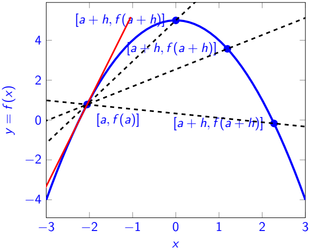
Chapter 3.1 - Tangent Lines and the Derivative at a Point

Instantaneous Rate of Change



$$f'(a) = \underbrace{\lim_{b \to a} \overbrace{\left(\frac{f(b) - f(a)}{b - a} \right)}^{\text{average rate}}}_{\text{instantaneous rate}}$$

$$f'(a) = \underbrace{\lim_{h \to 0} \overbrace{\left(\frac{f(a+h) - f(a)}{h} \right)}^{\text{average rate}}}_{\text{instantaneous rate}}$$

$f'(a)$ is the instantaneous rate of change of the function $f(x)$ at $x = a$. It is also

- $f'(a)$ also referred to as the derivative of $f(x)$ at $x = a$.
- The slope of the graph of $y = f(x)$ at $x = x_0$
- The slope of the tangent to the curve $y = f(x)$ at $x = x_0$
- The rate of change of $f(x)$ with respect to x at $x = a$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Example: For the quadratic function $2x^2 + 3$ find $f'(a)$ for any a.

$f'(a) =$

$$\lim_{h \to 0} \frac{2(a+h)^2 + 3 - (2a^2 + 3)}{h} =$$

$$\lim_{h \to 0} \frac{2a^2 + 4ah + h^2 + 3 - 2a^2 - 3}{h} =$$

$$\lim_{h \to 0} \frac{4ah + 2h^2}{h} = \lim_{h \to 0} 4a + 2h = 4a$$

Example: For $f(x) = |x|$, what is $f'(0)$?

$$\lim_{h \to 0^+} \frac{|0+h| - |0|}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1$$

$$\lim_{h \to 0^-} \frac{|0+h| - |0|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1$$

The tangent line at $x = a$ to $f(x)$ is the line which best approximates $f(x)$ near $x = a$ (in other words the line we see when we zoom in).

- $(a, f(a))$ is a point on the line.
- $f'(a)$ is the slope of the line.

Common ways to write the tangent line:

$$\begin{aligned} y - f(a) &= f'(a)(x-a) \\ y &= f(a) + f'(a)(x-a) \\ y &= \underbrace{f'(a)}_{=m} x + \underbrace{(f(a) - af'(a))}_{=b} \end{aligned}$$

Example: Find the tangent line to $y = \dfrac{1}{\sqrt{x}}$ at $x = 4$.

$$\begin{aligned} f'(4) &= \lim_{h\to 0} \frac{f(4+h) - f(4)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{\sqrt{4+h}} + \frac{1}{\sqrt{4}}}{h} \\ &= \lim_{h\to 0} \frac{\frac{2-\sqrt{4+h}}{2\sqrt{4+h}}}{h} \\ &= \lim_{h\to 0} \frac{4-4-h}{h2\sqrt{4+h}(2+\sqrt{4+h})} \\ &= \frac{-1}{2\sqrt{4}(2+\sqrt{4})} = -\frac{1}{16} \end{aligned}$$

$$y - \tfrac{1}{2} = -\tfrac{1}{16}(x-4)$$

$$y = -\tfrac{1}{16}x + \tfrac{3}{4}$$

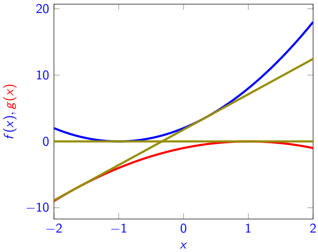
Example

Find *all* lines which are tangent to both
$f(x) = 2x^2 + 4x + 2$ and $g(x) = -x^2 + 2x - 1$.

Tangent to $f(x)$: $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} =$
$\lim_{h\to 0} \frac{2(a+h)^2+4(a+h)+2-2a^2-4a-2}{h} =$
$\lim_{h\to 0} \frac{4ah+2h^2+4h}{h} = \lim_{h\to 0} 4a+4+2h = 4a+4$
$y = (4a+4)x + (f(a) - a(4a+4))$
$y = (4a+4)x + (2a^2 + 4a + 2 - 4a^2 - 4)$
$y = (4a+4)x + (-2a^2 + 2)$

Tangent to $g(x)$: $g'(c) = \lim_{h\to 0} \frac{g(c+h)-f(c)}{h} =$
$\lim_{h\to 0} \frac{-(c+h)^2+2(c+h)-1+2c^2-2c+1}{h} =$
$\lim_{h\to 0} \frac{-2ch-h^2+2h}{h} = \lim_{h\to 0} -2c+2-h =$
$-2c+2$
$y = (-2c+2)x + (g(c) - c(-2c+2))$
$y = (-2c+2)x + (-c^2 + 2c - 1 + 2c^2 - 2c$
$y = (-2c+2)x + (c^2 - 1)$

The tangents are identical: $-2c+2 = 4a+4$
Hence $c = -2a - 1$.



$y = (4a+4)x + (-2a^2+2) \qquad y = (-2c+2)x + (c^2-1) \qquad c = -2a-1.$

Next $(-2a^2+2) = (c^2-1)$

$-2a^2+2 = (-2a-1)^2 - 1$

$-2a^2+2 = 4a^2 + 4a$

$0 = 6a^2 + 4a - 2$

$0 = 3a^2 + 2a - 1$

$0 = (a+1)(3a-1)$

Hence $a = -1$ or $a = \frac{1}{3}$

$y = (4(-1)+4)x + (-2(-1)^2+2) = 0$

$y = (4(\frac{1}{3})+4)x + (-2(\frac{1}{3})^2+2) = \frac{16}{3}x + \frac{16}{9}$

Note: One should check that it indeed works...

Chapter 3.1 Recap

- $f'(a)$ is the instantaneous rate of change of the function $f(x)$
- $f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$
- $f'(a)$ does not have to exist
- Tangent line at a: $y - f(a) = f'(a)(x - a)$