Chapter 3.2 - The Derivative as a Function

Recall The Derivative at a Point

The *derivative of a function* f *at a point* x_0, denoted $f'(x_0)$, is

$$\boxed{f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}}$$

provided this limit exists.

$f'(x_0)$ can be interpreted as

- The slope of the graph of $y = f(x)$ at $x = x_0$
- The slope of the tangent to the curve $y = f(x)$ at $x = x_0$
- The rate of change of $f(x)$ with respect to x at $x = x_0$

The Derivative of f

Try to compute $f'(x_0)$ for all x_0 at once.

The *derivative* of a function $f(x)$ is a function f' defined as

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

provided this limit exists.

Alternatively, making the change of variables $z = x + h$:

$$\boxed{f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}}$$

f is *differentiable* if the derivative is defined for all x

Example

Use $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ to compute the derivative of $f(x) = x^2$

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{2hx + h^2}{h} \\ &= \lim_{h \to 0} 2x + h = 2x \end{aligned}$$

Use $f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$ to compute the derivative of $f(x) = x^2$

$$\begin{aligned} f'(x) &= \lim_{z \to x} \frac{z^2 - x^2}{z - x} \\ &= \lim_{z \to x} \frac{(z-x)(z+x)}{z - x} \\ &= \lim_{z \to x} z + x = 2x. \end{aligned}$$

Example 2

Use $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ to compute the derivative of $g(t) = \sqrt{t}$

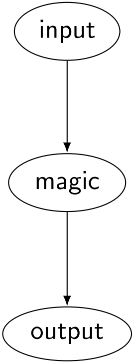
$$\begin{aligned} g'(t) &= \lim_{h\to 0} \frac{\sqrt{t+h}-\sqrt{t}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{t+h}-\sqrt{t}}{h} \cdot \frac{\sqrt{t+h}+\sqrt{t}}{\sqrt{t+h}+\sqrt{t}} \\ &= \lim_{h\to 0} \frac{t+h-t}{h[\sqrt{t+h}+\sqrt{t}]} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{t+h}+\sqrt{t}} \\ &= \frac{1}{\sqrt{t}+\sqrt{t}} = \frac{1}{2\sqrt{t}} \end{aligned}$$

Use $f'(x) = \lim_{z\to x} \dfrac{f(z)-f(x)}{z-x}$ to compute the derivative of $h(r) = \dfrac{1}{r}$

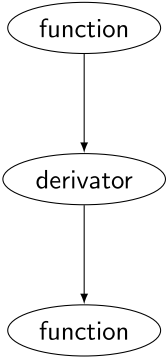
$$\begin{aligned} h'(r) &= \lim_{z\to r} \frac{\frac{1}{z}-\frac{1}{r}}{z-r} \\ &= \lim_{z\to r} \frac{\frac{1}{z}-\frac{1}{r}}{z-r} \cdot \frac{zr}{zr} \\ &= \lim_{z\to r} \frac{r-z}{(z-r)zr} \\ &= \lim_{z\to r} \frac{-1}{zr} \\ &= -\frac{1}{r^2} \end{aligned}$$

Function and Operator

Function



Derivative Operator



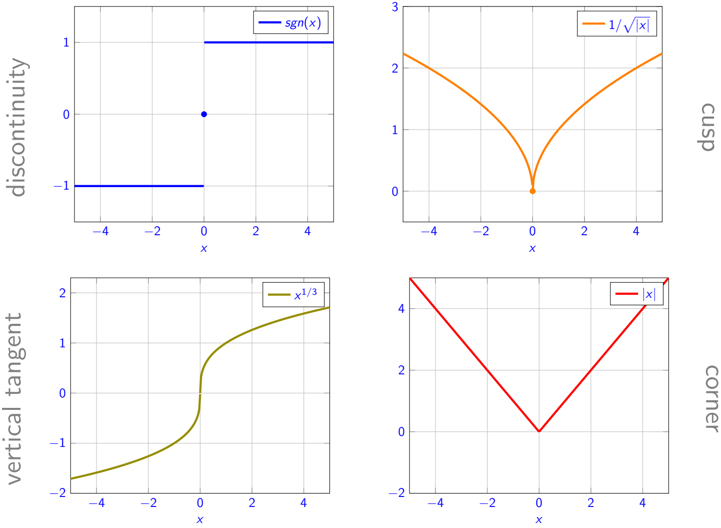
There are many ways to denote the derivative of $y = f(x)$.

Here's some common alternative notations:

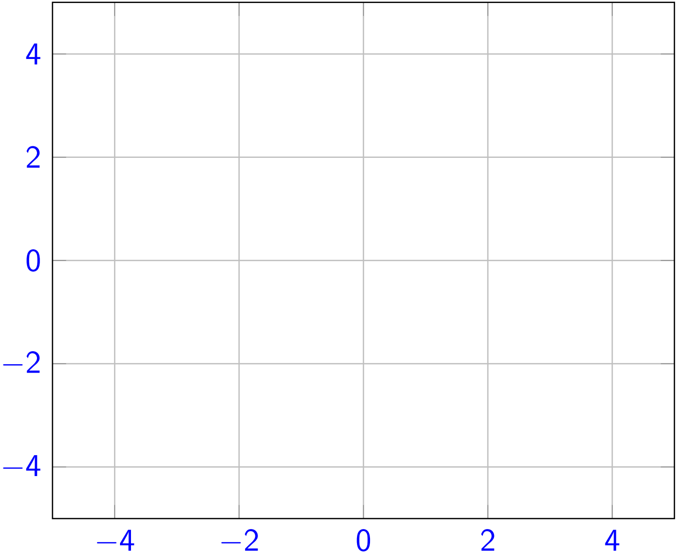
$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}\Big[f(x)\Big] = D(f)(x) = D_x[f(x)]$$

Where Derivative Does NOT Exists

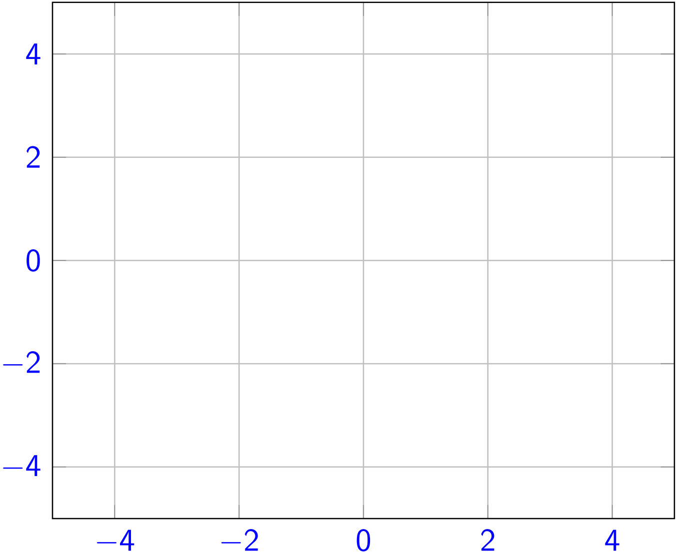
Derivative not existing is like tangent not existing.



Graphing the Derivative



Graphing f from f'



Continuity and Derivative

Theorem (Differentiability Implies Continuity)

If f has a derivative at $x = a$ then f is continuous at a.

Proof: Suppose that f is differentiable at $x = a$, then

$$\begin{aligned}
\lim_{x \to a} f(x) &= \lim_{x \to a} [f(x) - f(a) + f(a)] \\
&= \lim_{x \to a} \left[\frac{f(x) - f(a)}{x - a} \cdot (x - a) + f(a) \right] \\
&= f'(a) \cdot 0 + f(a) = f(a)
\end{aligned}$$

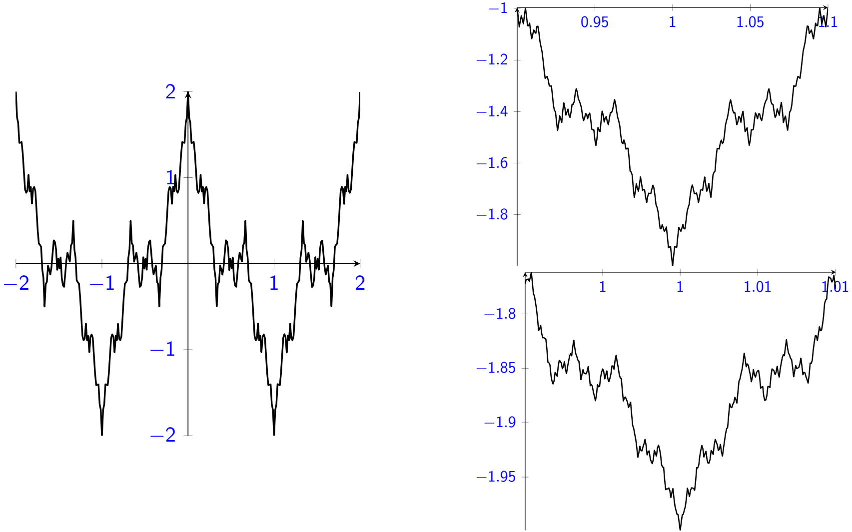
Note that the order matters here: if differentiable, then continuous.

The *converse* of this statement is *not true*!

There are very scary continuous function that are *differentiable nowhere*.

Most functions are actually very scary!

Weierstrass function $f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$



Looks like a fractal. Zooming in is NOT getting f closer to a line.

One-sided Derivatives

Recall: Limit exists if both one-sided limit exists and are equal.
Useful if the derivative does not exist, such as on the boundary of the domain.

Example: Compute one-sided derivative of $f(x) = |x|$ at $x_0 = 0$

From the left:

$$\boxed{\lim_{h \to 0^-} \frac{f(x_0 + h) - f(x_0)}{h}}$$

$$= \lim_{h \to 0^-} \frac{f(0 + h) - f(0)}{h}$$
$$= \lim_{h \to 0^-} \frac{|0 + h| - |0|}{h}$$
$$= \lim_{h \to 0^-} \frac{-h}{h} = -1$$

From the right:

$$\boxed{\lim_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h}}$$

$$= \lim_{h \to 0^+} \frac{f(0 + h) - f(0)}{h}$$
$$= \lim_{h \to 0^+} \frac{|0 + h| - |0|}{h}$$
$$= \lim_{h \to 0^+} \frac{h}{h} = 1$$

Chapter 3.2 Recap

- Derivative of f is a function whose values are slopes of tangents to f
- $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
- Derivative does not have to exists
- One sided version of derivative