Chapter 3.4: The Derivative as a Rate of Change

Example: For a circle what is the rate of change of the area with respect to the radius?

Let radius be r. Area is πr^2. We look at the area as a function of r. That is

$$f(r) = \pi r^2.$$

The instantaneous rate of change at r_0 is the same as the slope of a tangent line at r_0 to $f(r)$ and that is the same as the derivative of $f(r)$ at r_0. So the rate of change of the area is

$$\frac{d}{dr}[f(r)] = \frac{d}{dr}\left[\pi r^2\right] = 2\pi r.$$

Example: For a sphere what is the rate of change of the volume with respect to the radius?

The only difference from the circle is the formula for volume. Let r be the radius. Volume of a sphere of radius r is $V(r) = \frac{4}{3}\pi r^3$. Then the rate of change is

$$\frac{d}{dr}[V(r)] = \frac{d}{dr}\left[\frac{4}{3}\pi r^3\right] = 4\pi r^2.$$

Physics Basics

Object is moving with time t.

$s(t) =$ position (at time t)

$$v(t) = \text{velocity} = \text{how position changes} = s'(t)$$

$|v(t)| =$ speed

$$a(t) = \text{acceleration} = \text{how velocity changes} = v'(t) = s''(t)$$

What are the units?

Example: A cannon ball is launched straight into the air and its vertical position is given by $s = 200t - 20t^2$.

1. Compute v as a function of t

   $v = s' = 200 - 40t$

2. Compute a as a function of t

   $a = v' = -40$

3. What is the maximum height the ball obtains?

   When $v = 0$. So solving $0 = 200 - 40t$ gives $t = 5$. Max hight will be $s(5) = 200(5) - 20(25) = 500$.

4. What is v of the ball when it is 320 ft above the ground and heading downward?

   $320 = s = 200t - 20t^2 \quad 16 = 10t - t^2$

   $0 = t^2 - 10t + 16 \qquad 0 = (t-2)(t-8)$

   Max height is at $t = 5$, and so $t = 8$ must be downward. $v(8) = -120$.

1-D World

Example: Consider a particle moving along the y–axis, whose position is given by $s = t^3 - 6t^2 + 9t$

1. Find the particles velocity, speed, and acceleration as a function of t.

   $$v = s' = 3t^2 - 12t + 9 \quad \text{speed} = |v| = |3t^2 - 12t + 9| \quad a = v' = 6t - 12$$

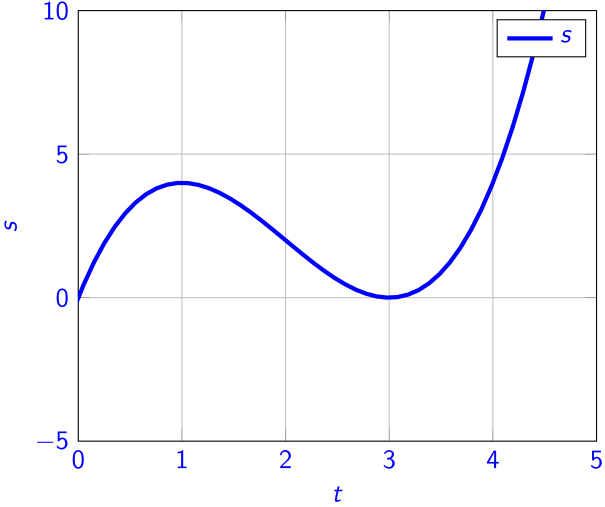
2. Find the particles displacement from $t = 0$ to $t = 2$.
   Displacement is the change in position.
   The displacement is $s(2) - s(0) = 2$
3. Find the particles average velocity from $t = 0$ to $t = 2$.
   The particle moved from $s(0)$ to (2) in time 2 so the average velocity is

   $$\frac{s(2) - s(0)}{2 - 0} = \frac{2 - 0}{2} = 1$$

4. Find the total distance the particle travels from $t = 0$ to $t = 2$.
   We need to be *very careful* with total distance traveled.
   The answer is **NOT** 2.

1-D World: Particle Path

Example: Consider a particle moving along the y–axis, whose position is given by $s = t^3 - 6t^2 + 9t$. Sketch v and a.



1-D World:

Example: Consider a particle moving along the y–axis, whose position is given by $s = t^3 - 6t^2 + 9t$

1. Find the particles velocity, speed, and acceleration as a function of t.

   $v = s' = 3t^2 - 12t + 9 \quad \text{speed} = |v| = |3t^2 - 12t + 9| \quad a = v' = 6t - 12$

4. Find the total distance the particle travels from $t = 0$ to $t = 2$. Let us find where the velocity is zero:

   $$0 = 3t^2 - 12t + 9 = 3(t^2 - 4t + 3) = 3(t-3)(t-1)$$

   Note that the velocity is positive from $t = 0$ to $t = 1$, and then negative from $t = 1$ to $t = 2$. This means that the particle *changed direction*. Let us compute the displacements on these time intervals:

   $$s(1) - s(0) = 4 - 0 = 4 \qquad s(2) - s(1) = 2 - 4 = -2$$

   Note that, as predicted, we moved in the negative direction from 1 to 2. To get the total distance travel, we need to ignore the "$-$" and add the two quantities together:

   $$4 + 2 = 6.$$

Chapter 3.4 Recap

- $s(t)$ is location as a function of time t
- $v(t)$ is velocity as a function of time t
- $a(t)$ is acceleration as a function of time t
- speed is $|v(t)|$
- $s' = v$
- $s'' = v' = a$