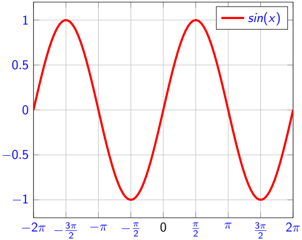
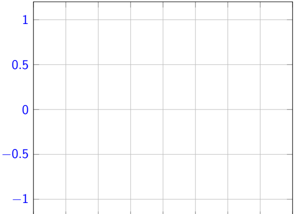
Chapter 3.5: Derivatives of Trigonometric Functions

Derivative of $\sin(x)$

$$\sin(x+h) = \sin(x)\cos(h) + \cos(x)\sin(h)$$

$$\lim_{h\to 0} \frac{\sin(h)}{h} = 1 \qquad\qquad \lim_{h\to 0} \frac{\cos(h)-1}{h} = 0$$





$$\frac{d}{dx}\Big[\sin(x)\Big]$$

$$= \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}$$

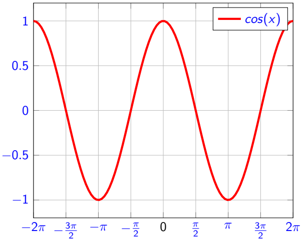
$$= \lim_{h\to 0} \sin(x)\frac{\cos(h)-1}{h} + \cos(x)\frac{\sin(h)}{h}$$

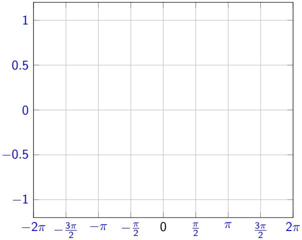
$$= \sin(x)\cdot 0 + \cos(x)\cdot 1$$

$$= \cos(x)$$

Derivative of $\cos(x)$

$$\cos(x+h) = \cos(x)\cos(h) - \sin(x)\sin(h)$$





$$\frac{d}{dx}\Big[\cos(x)\Big]$$

$$= \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h}$$

$$= \lim_{h\to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h}$$

$$= \lim_{h\to 0} \cos(x)\frac{\cos(h) - 1}{h} - \sin(x)\frac{\sin(h)}{h}$$

$$= \cos(x)\cdot 0 - \sin(x)\cdot 1$$

$$= -\sin(x)$$

Examples for sin and cos

$$\frac{d}{dx}[\sin(x)] = \cos(x) \qquad \frac{d}{dx}[\cos(x)] = -\sin(x)$$

1. $y = 3/x + 5\sin(x)$
   $y' = -3/x^2 + 5\cos(x)$

2. $f(x) = e^x \sin(x)$
   $f'(x) = e^x \sin(x) + e^x \cos(x)$

3. $f(x) = \sin(x)\cos(x)$
   $f'(x) = \cos(x)\cos(x) + \sin(x)\cdot(-\sin(x)) = \cos^2(x) - \sin^2(x) = \cos(2x)$

4. $\dfrac{d}{dx}\left[\dfrac{\cos(x)}{1-\sin(x)}\right]$

$$= \frac{-\sin(x)(1-\sin(x)) - \cos(x)(0-\cos(x))}{[1-\sin(x)]^2} = \frac{1-\sin(x)}{[1-\sin(x)]^2} = \frac{1}{1-\sin(x)}$$

Higher Derivatives

$$\frac{d}{dx}[\sin(x)] = cos(x) \qquad \frac{d}{dx}[\cos(x)] = -sin(x)$$

Find $\dfrac{d^{999}}{dx^{999}}\Big[\sin(x)\Big]$

Rewriting the derivative from above, we find that

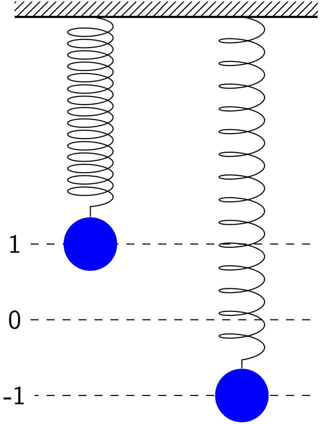
$$\frac{d}{dx}\Big[\sin(x)\Big] = \cos(x),\ \frac{d^2}{dx^2}\Big[\sin(x)\Big] = -\sin(x),\ \frac{d^3}{dx^3}\Big[\sin(x)\Big] = -\cos(x)$$
$$\frac{d^4}{dx^4}\Big[\sin(x)\Big] = \sin(x),\ \frac{d^5}{dx^5}\Big[\sin(x)\Big] = \cos(x),\ \frac{d^6}{dx^6}\Big[\sin(x)\Big] = -\sin(x)$$

and so on and so forth. Note that $999 = 996 - 3$, and since 996 is divisible by 4, it follows that

$$\frac{d^{996}}{dx^{996}}\Big[\sin(x)\Big] = \sin(x) \text{ hence } \frac{d^{999}}{dx^{999}}\Big[\sin(x)\Big] = \frac{d^3}{dx^3}\Big[\sin(x)\Big] = -\cos(x)$$

Simple Harmonic Motion

A weight on a spring is released from height one. The position of weight in time t is given by $s(t) = cos(t)$.



$v(t) = -sin(t)$

$a(t) = -cos(t)$

Example: In a damped motion the position is given by $s(t) = e^{-t}\cos(t)$. Compute the limit of acceleration as $t \to \infty$.

$$v(t) = \frac{d}{dx}\left[\frac{\cos(t)}{e^t}\right] = \frac{-\sin(t)e^t - \cos(t)e^t}{e^{2t}}$$

$$= \frac{-\sin(t) - \cos(t)}{e^t}$$

$$a = \frac{d}{dx}[v(t)] = \frac{d}{dx}\left[\frac{-\sin(t) - \cos(t)}{e^t}\right]$$

$$= \frac{(-\cos(t) + \sin(t))e^t - (-\sin(t) - \cos(t))e^t}{e^{2t}}$$

$$= 2\sin(t)e^{-t}$$

$$\lim_{t\to\infty} -2e^{-t} \le \lim_{t\to\infty} 2\sin(t)e^{-t} \le \lim_{t\to\infty} 2e^{-t}$$

$$0 \le \lim_{t\to\infty} 2\sin(t)e^{-t} \le 0$$

Other Trigonometric Functions

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \qquad \cot(x) = \frac{\cos(x)}{\sin(x)} \qquad \sec(x) = \frac{1}{\cos(x)} \qquad \csc(x) = \frac{1}{\sin(x)}$$

$$\frac{d}{dx}\Big[\tan(x)\Big] = \frac{d}{dx}\left[\frac{\sin(x)}{\cos(x)}\right] = \frac{\cos(x)\cos(x) + \sin(x)\sin(x)}{[\cos(x)]^2} = \frac{1}{[\cos(x)]^2} = \sec^2(x)$$

$$\frac{d}{dx}\Big[\cot(x)\Big] = \frac{d}{dx}\left[\frac{\cos(x)}{\sin(x)}\right] = \frac{-\sin(x)\sin(x) + \cos(x)\cos(x)}{[\sin(x)]^2} = \frac{-1}{[\sin(x)]^2} = -\csc^2($$

$$\frac{d}{dx}\Big[\sec(x)\Big] = \frac{d}{dx}\left[\frac{1}{\cos(x)}\right] = \frac{0\cos(x) + 1\sin(x)}{[\cos(x)]^2} = \frac{\sin(x)}{[\cos(x)]^2} = \sin(x)\sec^2(x)$$

$$\frac{d}{dx}\Big[\csc(x)\Big] = \frac{d}{dx}\left[\frac{1}{\sin(x)}\right] = \frac{0\sin(x) - 1\cos(x)}{[\sin(x)]^2} = \frac{-\cos(x)}{[\sin(x)]^2} = -\cos(x)\csc^2(x)$$

Chapter 3.5 Recap

- $$\frac{d}{dx}[\sin(x)] = cos(x)$$
- $$\frac{d}{dx}[\cos(x)] = -sin(x)$$
- Derivatives of other trigonometric functions follow from sin and cos.