Ore and Chvátal-type Degree Conditions for Bootstrap Percolation from Small Sets

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Abstract

Bootstrap percolation is a deterministic cellular automaton in which vertices of a graph G begin in one of two states, "dormant" or "active". Given a fixed integer r, a dormant vertex becomes active if at any stage it has at least r active neighbors, and it remains active for the duration of the process. Given an initial set of active vertices A, we say that G r-percolates (from A) if every vertex in G becomes active after some number of steps. Let $m(G,r)$ denote the minimum size of a set A such that G r-percolates from A.

Bootstrap percolation has been studied in a number of settings, and has applications to both statistical physics and discrete epidemiology. Here, we are concerned with degree-based density conditions that ensure $m(G,2)=2$. In particular, we give an Ore-type degree sum result that states that if a graph G satisfies $\sigma_2(G) \geq n-2$, then either $m(G,2)=2$ or G is in one of a small number of classes of exceptional graphs. We also give a Chvátal-type degree condition: If G is a graph with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$ such that $d_i \geq i+1$ or $d_{n-i} \geq n-i-1$ for all $1 \leq i < \frac{n}{2}$, then $m(G,2)=2$ or G falls into one of several specific exceptional classes of graphs. Both of these results are inspired by, and extend, an Ore-type result in [D. Freund, M. Poloczek, and D. Reichman, Contagious sets in dense graphs, to appear in *European J. Combin.*]

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1 Introduction

Bootstrap percolation, also known as the *irreversible r-threshold process* [17, 28] is a deterministic cellular automaton first introduced by Chalupa, Leath, and Reich [13]. Vertices of a graph are in one of two states, "dormant" or "active." Given an integer r, a dormant vertex becomes active only if it is adjacent to at least r active vertices. Once a vertex is activated, it remains in that state for the remainder of the process.

More formally, consider a graph G and let A denote the initial set of active vertices. For a fixed $r \in \mathbb{N}$, the *r-neighbor bootstrap percolation* process on G occurs recursively by setting $A = A_0$ and for each time step $t \geq 0$,

$$A_t = A_{t-1} \cup \{v \in V(G) : |A_{t-1} \cap N(v)| \geq r\},$$

where $N(v)$ denotes the neighborhood of the vertex v. If all of the vertices of G eventually become active, regardless of order, then we say that A is *r-contagious* or that G *r-percolates from* A. Given G and r, let $m(G, r)$ denote the minimum size of an r-contagious set in G. (Observe that $m(G, r) \geq \min\{r, |V(G)|\}$.)

Originally, bootstrap percolation was studied on lattices by statistical physicists as a model of ferromagnetism [13], and it can also be viewed as a model of discrete epidemiology, wherein a virus or other contagion is being transmitted across a network (cf. [7, 28]). (In the latter context, each vertex is either "infected" or "uninfected".) Further applications include the spread of influence in social networks [14, 22] and market stability in finance [2].

Much attention has been devoted to examining percolation in a probabilistic setting, referred to in [7] as the *random disease problem*. In this setting, the initial activated set A is selected according to some probability distribution. The parameter of interest is then the probability that G r-percolates from A, and in particular determining the threshold probability p for which G almost surely does (or does not) r-percolate when vertices are placed in A independently with probability p. Results have been obtained in this setting for a number of families of graphs, including random regular graphs [8], the Erdős–Rényi random graph $G_{n,p}$ [18, 21], hypercubes [3], trees [6], and grids [1, 4, 5].

In addition, there has recently been some interest in extremal problems concerning percolation in various families of graphs [10, 26, 27].

The problem has also been studied from the point of view of computational complexity. For $r \geq 3$, determining $m(G, r)$ is NP-complete [17], and determining $m(G, 2)$ is NP-complete even for graphs with maximum degree 4 [23]. Furthermore, it is computationally difficult to approximate $m(G, r)$ to within a polylogarithmic factor unless NP has quasi-polynomial algorithms [14]. Notice that $m(G, 1)$ is always equal to the number of connected components of G.

1.1 Degree-Based Results

In this paper, we are interested in degree-based density conditions that ensure that a graph G will percolate from a small set of initially activated vertices. Freund, Poloczek, and Reichman [19] showed that for each $r \geq 2$, if G has order n and $\delta(G) \geq \frac{r-1}{r}n$, then $m(G, r) = r$. Note that when $r = 2$, this is the same as Dirac's condition for hamiltonicity [16]. Recently, Gunderson [20]

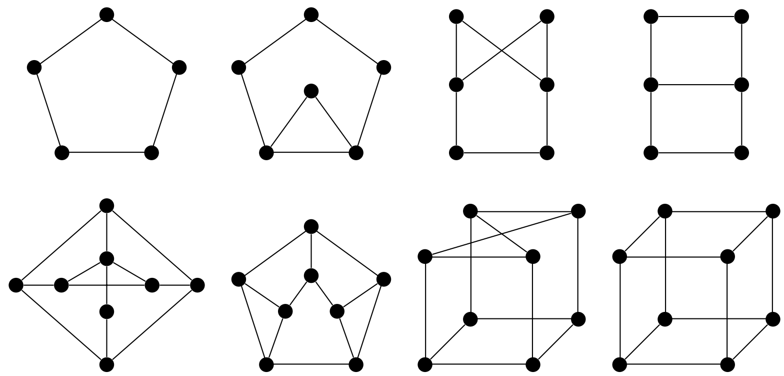

Figure 1: The family $\mathcal{X}$: Small exceptional graphs for Theorem 2.

showed that if $n \geq 13$ and $\delta(G) \geq n/2 + 1$, then $m(G, 3) = 3$, and that for each $r \geq 4$, if n is sufficiently large and $\delta(G) \geq n/2 + r - 3$, then $m(G, r) = r$. Moreover, both bounds are sharp.

Let $\sigma_2(G)$ denote the minimum degree sum of a pair of nonadjacent vertices in a graph G. Ore [25] proved that every graph G of order $n \geq 3$ that satisfies $\sigma_2(G) \geq n$ is hamiltonian. Freund, Poloczek, and Reichman [19] also showed that Ore's condition is sufficient to ensure that a graph 2-percolates from the smallest possible initially activated set.

Theorem 1 ([19])**.** *Let $n \geq 2$. If G is a graph of order n and $\sigma_2(G) \geq n$, then $m(G, 2) = 2$.*

Note that hamiltonicity alone is not sufficient to conclude that a graph G satisfies $m(G, 2) = 2$, as $m(C_n, 2)$ tends to infinity with n. Rather, Theorem 1 is part of a diverse collection of results that demonstrate that many sufficient density conditions for hamiltonicity imply a much richer structure that allows for stronger conclusions (cf. [11, 12]).

In this paper, we improve Theorem 1 in several ways. First, we characterize graphs of order n with $\sigma_2 \geq n - 2$ and $m(G, 2) > 2$. These will consist of four infinite families of graphs $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ and a finite set of graphs $\mathcal{X}$. The graphs in $\mathcal{X}$ are depicted in Figure 1.

The class $\mathcal{G}_0$ consists of all graphs which are unions of two disjoint non-empty cliques X, Y. Note that X and Y can be of different sizes. Graphs in $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ are formed from $\mathcal{G}_0$ by selecting $\{x, x'\} \subseteq X$, $\{y, y'\} \subseteq Y$, adding the edges xy, $x'y'$, and deleting the edges xx' and yy' if they exist (see Figure 2). For simplicity, we have distinguished the cases where $x = x'$ and $y = y'$ ($\mathcal{G}_1$), $x \neq x'$ and $y \neq y'$ ($\mathcal{G}_2$) and $x = x'$ and $y \neq y'$ ($\mathcal{G}_3$). It is easy to see that any graph $G \in \mathcal{G}_0 \cup \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$ containing at least one vertex in each of X and Y that is not adjacent to any vertex in the other set has $\sigma_2(G) = |V(G)| - 2$ and $m(G, 2) > 2$.

Theorem 2. *Let G be a graph of order $n \geq 2$ such that G is not in $\mathcal{G}_0$, $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$ or $\mathcal{X}$. If $\sigma_2(G) \geq n - 2$, then $m(G, 2) = 2$.*

In particular, Theorem 2 implies that C_5 is the only graph G with $\sigma_2(G) = |V(G)| - 1$ and $m(G, 2) > 2$.

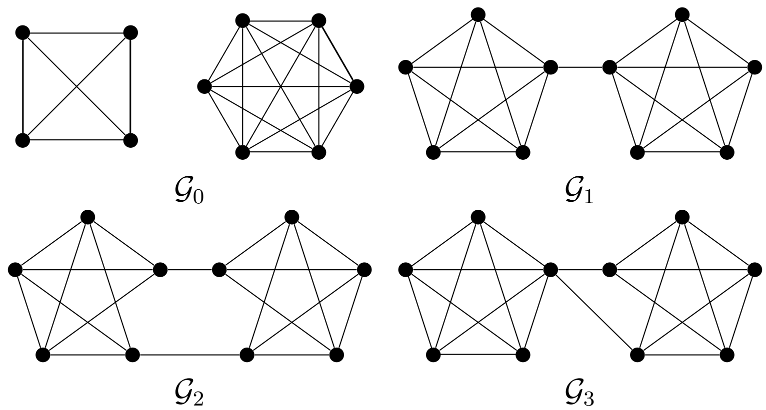

Figure 2: Examples of graphs in $\mathcal{G}_0$, $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$ for $n = 10$.

Second, we prove a degree sequence condition for $m(G, 2) = 2$. Let G be a graph with degree sequence $d_1 \leq \cdots \leq d_n$. We say that G satisfies *Chvátal's condition* if

$$d_i \geq i + 1 \quad \text{or} \quad d_{n-i} \geq n - i, \quad \forall i, 1 \leq i < \tfrac{n}{2}. \tag{1.1}$$

In [15], Chvátal proved that a graph G of order $n \geq 3$ that satisfies Chvátal's condition is hamiltonian.

Here, we show that a slightly weaker Chvátal-type condition implies that $m(G, 2) = 2$. We say that a graph G satisfies the *weak Chvátal condition* if

$$d_i \geq i + 1 \quad \text{or} \quad d_{n-i} \geq n - i - 1, \quad \forall i, 1 \leq i < \tfrac{n}{2}, \tag{1.2}$$

and prove the following.

Theorem 3. *If G is a graph with degree sequence $d_1 \leq \cdots \leq d_n$ that satisfies the weak Chvátal condition (1.2), then either $m(G, 2) = 2$ or one of the following holds:*

- *G is disconnected,*
- *G contains two vertices of degree one and $G \notin \{P_2, P_3\}$, or*
- *G is C_5.*

We denote the path on k vertices by P_k and the cycle on k vertices by C_k. Much as Chvátal's Theorem implies Ore's Theorem for hamiltonicity, each of Theorems 2 or 3 implies Theorem 1.

1.2 Notation

Let G be a graph and let $U \subseteq V(G)$ and let $v \in V$. We denote by $G[U]$ the subgraph of G induced by U. The notation $\Delta(G)$ means the maximum degree of G and $N(v)$ is the set of neighbors of v. We denote by $N_U(v)$ the set of neighbors in U, that is $N(v) \cap U$. The notation $d(v)$ means the degree of v and $d_U(v)$ is $|N_U(v)|$.

2 Proof of Theorem 2

Proof. Let G be a graph of order n with $\sigma_2(G) \geq n-2$ that is not in one of the exceptional classes $\mathcal{X}$, $\mathcal{G}_0$, $\mathcal{G}_1$, $\mathcal{G}_2$, or $\mathcal{G}_3$. Throughout the proof, amongst all subsets of $V(G)$ that can be activated from a starting set of two vertices, let I (for "infected") have maximum size, and let $U = V(G) \setminus I$ denote the set of vertices that remain dormant from this starting set. We repeatedly use the following observation that follows from the maximality of I.

Observation 4. *Each vertex in U has at most one neighbor in I.*

Notice that our assumption on $\sigma_2(G)$ implies that $\Delta(G) \geq (n-2)/2$.

For $n \leq 11$, Nauty [24] was utilized to generate all graphs with $m(G, 2) > 2$, which is precisely the set $\mathcal{X}$. Thus, we may assume that $n \geq 12$ as we proceed. Further, if G is disconnected, the degree sum condition guarantees that G has exactly two complete components and so $G \in \mathcal{G}_0$, a contradiction. We may therefore assume that G is connected. The following sequence of claims establishes some important facts about the size and structure of I and U.

Claim 5. *$|I| > \frac{n}{2}$ and $G[U]$ is complete.*

Proof. First we show $|I| \geq 4$. Suppose for a contradiction that $|I| < 4$. If G contains a triangle T, then since G is connected some vertex $y \in T$ has a neighbor y' in $G - T$. Activating y' and some $x \in V(T) \setminus \{y\}$ results in at least 4 activated vertices.

If G contains a C_4, we can activate the entire cycle starting with either pair of nonadjacent vertices. Hence we may suppose that G contains neither a triangle nor C_4.

Let w be a vertex with $d(w) = \Delta(G) \geq (n-2)/2 \geq 5$. As G is triangle-free, $N(w)$ is independent. Let x, y and z be distinct neighbors of w and suppose that $d(x), d(y) \leq d(z)$. As G contains no C_4, $N(x)$, $N(y)$ and $N(z)$ intersect pairwise only in w, so that $d(x)+d(y)+d(z) \leq n-1$. This, however, is a contradiction, as then $d(x) + d(y) < n-2$, so we may assume that $|I| \geq 4$.

Next we establish that $|I| \geq |U|$. It suffices to show that $m(G[U], 2) = 2$, which would imply that $|U| \leq |I|$ since $|I|$ is maximum among all sets infected by 2 vertices. To that end, let u and v be nonadjacent vertices in U, and recall that every vertex in U has at most one neighbor in I. Consequently, as $|I| \geq 4$,

$$d_U(u) + d_U(v) \geq d(u) - 1 + d(v) - 1 \geq n - 4 \geq |U|.$$

Thus, $m(G[U], 2) = 2$ by Theorem 1, so $|U| \leq |I|$. Therefore, the first condition $|I| > \frac{n}{2}$ holds unless $|U| = |I| = \frac{n}{2}$. We will deal with this case after establishing $G[U]$ is complete.

Suppose $G[U]$ is not a complete graph, and let u and v be nonadjacent vertices in U. Then, as $|U| \leq \frac{n}{2}$,

$$d_U(u) + d_U(v) \leq 2(|U| - 1) \leq n - 2,$$

which is a contradiction unless equality holds. If equality holds, then U induces a complete graph on exactly $\frac{n}{2}$ vertices minus a matching M, where every vertex on M has a neighbor in I. Notice that in this case, $G[U]$ percolates from any choice of two vertices in U (since $\frac{n}{2} \geq 5$). Activating two neighbors of a vertex on M – one in I and one in U – results in at least $|U| + 1$ activated vertices, a contradiction to $|I| = \frac{n}{2}$ being maximum. Therefore, $G[U]$ must be a complete graph.

We finally return to the case where $|U| = |I| = \frac{n}{2}$. Let $v \in I$ have a neighbor u in U. For any z in $U \setminus \{u\}$, initially activating $\{v, z\}$ leads to (at least) the activation of $U \cup \{v\}$, contradicting the maximality of $|I|$ and establishing Claim 5. $\square$

Partition I into sets I_0 and I_1, where I_1 are those vertices with at least one neighbor in U, so that vertices in I_0 have no neighbors in U. Since $|I| > |U|$, and no vertex in U has more than one neighbor in I, there exists a vertex $w \in I_0$. Let $u \in U$ and observe that

$$n - 2 \le d(w) + d(u) \le |I| - 1 + |U| = n - 1.$$

Hence w has at most one non-neighbor in I, so that $G[I_0]$ is a complete graph minus a matching. Consequently, if any three vertices in I are activated, then all of I_0 will be activated in the following step of the percolation.

Claim 6. *Every vertex in U has exactly one neighbor in I.*

Proof. By Observation 4, it suffices to show that each vertex in U has at least one neighbor in I. Suppose otherwise, so that there exists $z \in U$ with no neighbors in I, and therefore there are at least two vertices w_1 and w_2 in I_0. Then

$$n - 2 \le d(w_i) + d(z) \le |I| - 1 + |U| - 1 = n - 2$$

for $i \in \{1, 2\}$. Hence $d(w_1) = d(w_2) = |I| - 1$ and w_1, w_2 are adjacent to all vertices of I. Let $v \in I$ and $u \in U$ be adjacent vertices. If we initially activate $\{w_1, u\}$, this in turn would activate at least $I \cup \{u\}$, contradicting the maximality of $|I|$. Consequently, every vertex in U has a neighbor in I, establishing Claim 6. $\square$

Claim 7. $|I_1| \ge 2$.

Proof. Suppose, towards a contradiction, that $\{v\} = I_1$, so that v is adjacent to all vertices in U. If x_1 and x_2 in $N_I(v)$ are adjacent, then for any $u \in U$, initially activating $\{x_1, u\}$ results in at least three vertices in I being activated. This ultimately results in the activation of at least $I \cup \{u\}$, contradicting the maximality of $|I|$.

Hence $N_I(v)$ is an independent set in I_0, so it has size at most two. Thus I_0 induces either a complete graph with exactly one vertex adjacent to v, or a complete graph with some edge ab deleted and v adjacent to only a and b. As U is a clique by Claim 5, we conclude that $G \in \mathcal{G}_1 \cup \mathcal{G}_3$, a contradiction. This proves Claim 7. $\square$

Claim 8. *There is some vertex in I_1 with at least two neighbors in U. Hence $|U| > |I_1|$.*

Proof. By Claims 6 and 7 we may assume that $|I_1| \ge 2$, and further, towards a contradiction, that the edges between I and U form a matching. Recall also that $G \notin \mathcal{G}_1 \cup \mathcal{G}_3$. If $p \in I_1$, then since $|I_1| \ge 2$ there is some vertex in U not adjacent to p, implying that $d(p) \ge n - 2 - |U| = |I| - 2$. Hence p, and therefore any vertex in I, has at most two non-neighbors in $I - \{v\}$. Since $n \ge 12$, we have $|I| \ge 7$, so that $d_I(p) \ge 5$. Further, as every vertex in I is nonadjacent to at most two other vertices in I, there is an edge w_1w_2 in $N_I(p)$, and there must be a vertex w_3 in I with at least two neighbors in $\{w_1, w_2, p\}$.

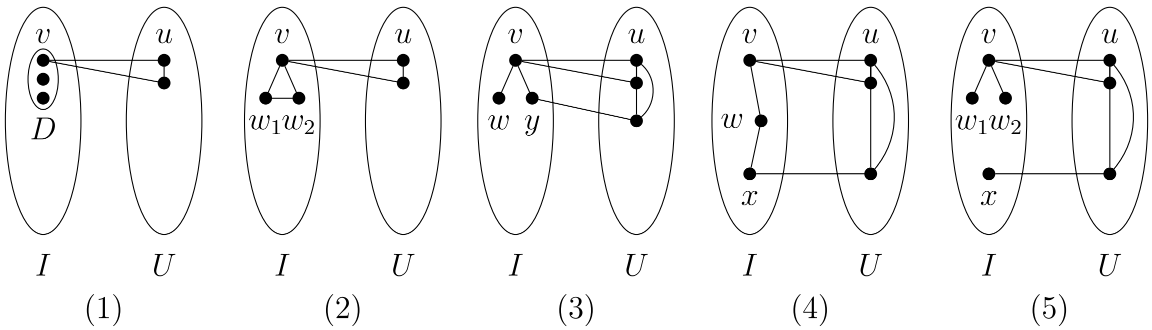

Figure 3: The situation in Claim 9.

Let $u \in N_U(p)$ and consider the percolation that occurs if the initially activated set is $\{w_1, u\}$. In the first step, p activates, followed by w_2 and w_3. This gives four active vertices $W = \{p, w_1, w_2, w_3\}$ in I. As every vertex in I has at most two non-neighbors in W, at a minimum $I \cup \{u\}$ becomes active, contradicting the maximality of $|I|$. This concludes the proof of Claim 8. $\square$

Claim 9. *Let $v \in I_1$ have at least two neighbors in U and let u be one such neighbor. Also, let D be a subset of I containing at least three vertices, including v, and let $x \in I_1 \setminus \{v\}$. The following hold:*

(1) There is no set of size 2 that activates $U \cup D$;

(2) $N_I(v)$ is an independent set;

(3) If there is a vertex y in $N_I(v) \cap I_1$, then y is the only neighbor of v in I;

(4) v and x have no common neighbor;

(5) $|N_I(v)| = 1$, $I_1 = \{v, x\}$ and x has exactly one neighbor in U.

Proof. Before we begin, it is useful to note that if u and v are activated, then so too will be all of U, as $G[U]$ is complete. Also, the proofs of *(1)–(5)* are illustrated in Figure 3.

(1): Suppose that $U \cup D$ can be activated starting from two vertices, a and b. Let $w \in I_0$. By Claim 7 and applying the bound on $\sigma_2(G)$ to w and u, the vertex w can have at most one non-neighbor in $G[I]$. Hence, w has at least two neighbors in D and so becomes activated as well. Thus, the set of vertices activated starting with $\{a, b\}$ contains $U \cup I_0 \cup \{v\}$. Since $|U| > |I_1|$ by Claim 8, we obtain a contradiction to the maximality of $|I| = |I_0 \cup I_1|$.

(2): Suppose otherwise, and let w_1 and w_2 be adjacent vertices in $N_I(v)$. Consider initially activating $\{w_1, u\}$. In the first three steps, all of $\{u, v, w_1, w_2\} \cup U$ is activated, contradicting *(1)* with $D = \{v, w_1, w_2\}$.

(3): Assume otherwise, that v has two neighbors y and w in I, where y has a neighbor in U. Initially activating $\{w, u\}$ activates v in the first step, and U in the step that follows. Consequently, y is activated, contradicting *(1)* with $D = \{v, w, y\}$.

(4): Let x be in $I_1 \setminus \{v\}$, as given, and assume that w is a common neighbor of v and x. Initially activating $\{w, u\}$ then activates v, x and the entirety of U in four iterations, again contradicting *(1)*.

(5): Suppose first that w_1 and w_2 are distinct neighbors of v in I. By *(2)*, *(3)* and *(4)*, they are nonadjacent, they have no neighbors in U, and neither is adjacent to x. Then $d(w_1) + d(u) \leq |I| - 3 + |U| = n - 3$, a contradiction. Hence $|N_I(v)| \leq 1$.

By Claims 5 and 8, $|I| - 1 \geq |U| \geq |I_1| + 1$. Hence $|I_0| \geq 2$ and there exists $w \in I_0 \setminus N(v)$. Applying $\sigma_2(G) \geq n - 2$ to nonadjacent vertices v and w we obtain

$$|I| - 2 + d(v) \geq d(w) + d(v) \geq n - 2 = |U| + |I| - 2.$$

Hence $|U| \leq d(v)$. On the other hand $d(v) \leq |U \setminus N(x)| + 1$, so $d(v) = |U|$. It follows that v is adjacent to all but one vertex of U, so that $I_1 = \{v, x\}$ and x has exactly one neighbor in U. This completes the proof of Claim 9. $\square$

Claim 9 implies that G is in $\mathcal{G}_2 \cup \mathcal{G}_3$, the final contradiction needed to complete the proof of Theorem 2. $\square$

3 Proof of Theorem 3

Proof. For $n \leq 12$, Theorem 3 was verified using Nauty [24], so throughout the proof, we may assume that $n \geq 13$. Suppose then that G is a graph of order n that satisfies the weak Chvátal condition (1.2). Further, by way of contradiction, suppose that $m(G, 2) > 2$ and that G is connected and has at most one vertex of degree 1. Also, let $V(G) = \{v_1, v_2, \ldots, v_n\}$, where $d(v_i) = d_i$ and $d_i \leq d_j$ whenever $i \leq j$, and let L be the set of vertices of degree at least $\frac{n-1}{2}$.

If G satisfies the full Chvátal condition (1.1), one of the following must hold:

- Type 1: $d(v) \geq \frac{n}{2}$ for all $v \in L$ and $|L| > \frac{n}{2}$, or
- Type 2: $d(v) > \frac{n}{2}$ for all $v \in L$ and $|L| \geq \frac{n}{2}$.

As one would expect, the weak Chvátal condition (1.2) results in slightly weaker conclusions. For even n, L is either Type 1 or

- Type 2^-: $d(v) \geq \frac{n}{2}$ for all $v \in L$ and $|L| \geq \frac{n}{2}$.

For odd n, L is either Type 2 or

- Type 1^-: $d(v) \geq \frac{n-1}{2}$ for all $v \in L$ and $|L| \geq \frac{n+1}{2}$.

In particular, notice that if n is even, then $d(v) \geq \frac{n}{2}$ for all $v \in L$, and if n is odd, then $|L| \geq \frac{n+1}{2}$.

Let I have maximum size among activated sets starting from two vertices and satisfying $I \cap L \neq \emptyset$. Furthermore, let $U = V \setminus I$, $L_I = L \cap I$ and $L_U = L \cap U$. Notice that Observation 4, that each vertex in U has at most one neighbor in I, holds in this proof as well.

Claim 10. $|I| \leq \frac{n}{2}$.

Proof. Suppose, towards a contradiction, that $|I| > \frac{n}{2}$, so that $|U| < \frac{n}{2}$. Let $u \in U$ and note that since u has at most one neighbor in I, $d(u) \leq |U|$, which implies that $d(v_{|U|}) \leq |U|$. The weak Chvátal condition (1.2) then implies that $d(v_{n-|U|}) \geq n - |U| - 1$. Let $X = \{v_{n-|U|}, \ldots, v_n\}$, so that $d(v) \geq n - |U| - 1 = |I| - 1 \geq |U|$ for all $v \in X$. Note $|X| = |U| + 1$. Up to relabeling the vertices, while maintaining the order of their degrees, we may assume that $X \subseteq I$.

Suppose that there are vertices v and w in X with no neighbors in U, implying that they have degree equal to $|I| - 1$ in I. Let $u \in U$ have a neighbor in I. Initially activating $\{v, u\}$ then activates w in the second round and consequently activates $I \cup \{u\}$, which contradicts the maximality of $|I|$. Thus there is at most one vertex $v \in X$ with no neighbors in U.

Since $|X| = |U| + 1$, v is the unique member of X that has no neighbors in U and every $w \in X \setminus \{v\}$ has exactly one neighbor in U. Let $w \in X \setminus \{v\}$ and let $u \in U$ be adjacent to w. Initially activating $\{v, u\}$ activates w in the first round, followed by all vertices in $I \cap N(w)$. By the maximality of $|I|$ and the fact that $|N(w) \cap I| \geq |I| - 2$, there is exactly one vertex $z \in I$ that is not adjacent to w. Moreover, z is adjacent only to v in I, otherwise it becomes active.

If z had a neighbor in U, then there would be two vertices in X with no neighbors in U. Hence z is a vertex of degree one. If u has a neighbor u' in U, then u' has a neighbor in X, and hence also activates since all of X is activated. This would contradict the maximality of $|I|$. Hence u is also a vertex of degree one, a contradiction to the assumption that G has at most one vertex of degree one. This concludes the proof of Claim 10. □

Claim 11. $|I| < \frac{n}{2}$.

Proof. Assume otherwise, then by Claim 10, $|I| = |U| = \frac{n}{2}$, which implies that n is even. If $u \in L_U$, then $d(u) \geq \frac{n}{2}$. Hence u is adjacent to all other vertices in U and has exactly one neighbor in I. If $|L_U| \geq 2$, let u, $x \in L_U$, let $v \in N_I(u)$ and initially activate $\{v, x\}$. It follows that all of U is activated by the end of the second round. Because this activates $U \cup \{v\}$, it contradicts the maximality of I.

Suppose then, that $|L_U| \leq 1$, so that $|L_I| \geq \frac{n}{2} - 1$. As n is even, every vertex in L has degree at least $\frac{n}{2}$. Hence every vertex in L_I has at least one neighbor in U. Since the number of edges between I and U is at most $\frac{n}{2}$, there is at most one vertex in L_I with more than one neighbor in U and all the other vertices of L_I are complete to I. Hence $G[I]$ is either a complete graph or a complete graph minus an edge, and all but at most one vertex in I has a neighbor in U.

As $|I| = \frac{n}{2} \geq 4$, there are vertices v and w in I such that both vertices are adjacent to all of I, save themselves, and v has a neighbor u in U. Activating u and w results in activating $I \cup \{u\}$, contradicting the maximality of I. This concludes the proof of Claim 11. □

Let
$$p = \frac{n}{2} - |I| \geq \frac{1}{2}.$$
Notice that p is an integer if n is even.

Claim 12. *$|L_U| \geq 3$.*

Proof. If $p \geq 3$, the claim follows from $|L_I| \leq |I| \leq \frac{n}{2} - 3$ and $|L_I| + |L_U| \geq \frac{n}{2}$. We assume for contradiction that $|L_U| \leq 2$. We distinguish the following three cases based on the parity of n and p being $\frac{1}{2}$.

Case 1: n is even and $p > \frac{1}{2}$.

Since n is even, for every $v \in L_I$, we have $d_U(v) \geq p + 1$. Every vertex in U has at most one neighbor in I, so $|U| \geq |L_I|(p+1)$. Since $|L_U| \leq 2$, we get $|L_I| \geq |L| - 2 \geq \frac{n}{2} - 2$. Therefore,
$$\frac{n}{2} - 2 \leq |L_I| \leq \frac{|U|}{p+1} = \frac{n/2 + p}{p+1}.$$
However, if $p \in \{1, 2\}$, this inequality fails, a contradiction.

Case 2: n is odd and $p > \frac{1}{2}$.

Since n is odd and $p > \frac{1}{2}$, for every $v \in L_I$, we have $d_U(v) \geq p + \frac{1}{2}$. Every vertex in U has at most one neighbor in I, so $|U| \geq |L_I|(p + 1/2)$. Since $|L| \geq \frac{n+1}{2}$, we obtain $|L_I| \geq \frac{n-3}{2}$. Therefore,
$$\frac{n-3}{2} \leq |L_I| \leq \frac{|U|}{p + 1/2} = \frac{n/2 + p}{p + 1/2}.$$
However, if $p \in \{\frac{3}{2}, \frac{5}{2}\}$, this inequality fails, a contradiction.

Case 3: $p = \frac{1}{2}$.

Since $p = \frac{1}{2}$, n is odd. In this case, $|I| = \frac{n-1}{2}$ and $|L_I| \geq |L| - 2 \geq \frac{n-1}{2} - 1$.

Every vertex in L_I has degree at least $\frac{n-1}{2}$, and therefore must have a neighbor in U. Since every vertex in U has at most one neighbor in I, there are at most $\frac{n+1}{2}$ edges between L_I and U. Hence there are at most two vertices in L_I with more than one neighbor in U. If there are two such vertices in L_I, then both have exactly two neighbors in U. If there is exactly such vertex in L_I, then it has at most three neighbors in U.

Consequently, I is a complete graph of order at least 5 with the exception either a single edge or two incident edges. As each vertex in L_I has at least one neighbor in U, it is straightforward to select two vertices that, when initially activated, activate all of I and at least one vertex in U. This concludes the proof of Claim 12. □

Claim 13. *L_U is a clique.*

Proof. Assume otherwise, and let u and v be nonadjacent vertices in L_U. We claim that initially activating u and v generates a contradiction to the maximality of I.

As every vertex in U has at most one neighbor in I, both u and v have at least $\frac{n-1}{2} - 1$ neighbors among the other $\frac{n}{2} + p - 2$ vertices of U. Let r be the number of vertices in U that are common neighbors of u and v. Counting edges from u and v, we obtain

$$\begin{aligned} d(u)-1+d(v)-1 &\leq 2r+(|U\setminus\{u,v\}|-r) \\ 2\left(\frac{n-1}{2}-1\right) &\leq 2r+\left(\frac{n}{2}+p-2-r\right) \\ \frac{n}{2}-p-1 &\leq r \end{aligned}$$

Together with $\{u,v\}$, a total of $r+2 \geq \frac{n}{2}-p+1 = |I|+1$ vertices become activated by the second round, contradicting the maximality of $|I|$ and proving Claim 13. □

Claim 14. *Vertices in L_U have no neighbors in I.*

Proof. Suppose for a contradiction that $v \in I$ and $u \in L_U$ are adjacent. Let w and z be two vertices in L_U aside from u, which exist by Claim 12. Initially activate the set $\{v,w\}$. In the first round, u activates, and in the second round, the remainder of L_U, including z, activates.

Counting edges from u, w, and z to the remainder of U, and letting r denote the number of vertices adjacent to at least two of u, w or z, we get

$$\begin{aligned} d(u)-3+d(z)-3+d(w)-3 &\leq 3r+(|U\setminus\{u,v,w\}|-r) \\ 3\left(\frac{n-1}{2}-3\right) &\leq 3r+\left(\frac{n}{2}+p-3-r\right) \\ n-p-7.5 &\leq 2r \\ \frac{n}{2}-\frac{p}{2}-3.75 &\leq r \end{aligned}$$

When we include u, v, w, and z, we see that there are at least $\frac{n}{2}-\frac{p}{2}+0.25$ activated vertices. This contradicts the maximality of $|I| = \frac{n}{2}-p$ and concludes the proof of Claim 14. □

To finish the proof, let u, w, $z \in L_U$ and initially activate the set $\{z,w\}$, so that u is activated in the first round. By counting edges from $\{u,w,z\}$ to $U\setminus\{u,w,z\}$, and again letting r denote the number of vertices in $U\setminus\{u,w,z\}$ that are adjacent to at least two vertices in $\{u,w,z\}$, we get

$$\begin{aligned} d(u)-2+d(z)-2+d(w)-2 &\leq 3r+(|U\setminus\{u,v,w\}|-r) \\ 3\left(\frac{n-1}{2}-2\right) &\leq 3r+\left(\frac{n}{2}+p-3-r\right) \\ n-p-4.5 &\leq 2r \\ \frac{n}{2}-\frac{p}{2}-2.75 &\leq r \end{aligned}$$

Together with u, w, and z, we get at least $\frac{n}{2}-\frac{p}{2}+0.25$ activated vertices, the final contradiction to the maximality of $|I|$ necessary to complete the proof of Theorem 3. □

Conclusion

Theorem 3 gives a sharp degree condition that ensures that a graph G satisfies $m(G,2) = 2$, in that it provides a class of graphs that demonstrates the sharpness of the weak Chvátal condition for this property. However, Chvátal-type conditions are often shown to be best possible in a different manner, which gives rise to a perhaps challenging open problem related to the work in this paper.

Let $S = (d_1, \ldots, d_n)$ and $S' = (d'_1, \ldots, d'_n)$ be sequences of real numbers. We say that S *majorizes* S', and write $S \succeq S'$, if $d_i \geq d'_i$ for every i. A sufficient degree condition $\mathcal{C}$ for a graph property $\mathcal{P}$ is *monotone best possible* if whenever $\mathcal{C}$ does not ensure every realization of a degree sequence π has property $\mathcal{P}$, there is some graphic sequence $\pi' \succeq \pi$ such that π' has a realization without property $\mathcal{P}$.

The Chvátal condition is a monotone best possible degree condition for hamiltonicity [15], and [9] is a thorough survey of monotone best possible degree criteria for a number of graph properties. However, it is easy to show that it is not the case that either the Chvátal condition (1.1) or the weak Chvátal condition (1.2) is monotone best possible for the property $m(G,2) = 2$. This gives rise to the following problem:

Problem 1. Determine a monotone best possible degree condition for the property "$m(G,2) = 2$".

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