Solving Turán's Tetrahedron Problem for the ℓ_2-Norm

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Abstract

Turán's famous tetrahedron problem is to compute the Turán density of the tetrahedron K_4^3. This is equivalent to determing the maximum ℓ_1-norm of the codegree vector of a K_4^3-free n-vertex 3-uniform hypergraph. We will introduce a new way for measuring extremality of hypergraphs and determine asymptotically the extremal function of the tetrahedron in our notion.

The codegree squared sum, $\text{co}_2(G)$, of a 3-uniform hypergraph G is the sum of codegrees squared $d(x,y)^2$ over all pairs of vertices xy, or in other words, the ℓ_2-norm of the codegree vector of the pairs of vertices. Define $\text{exco}_2(n,H)$ to be the maximum $\text{co}_2(G)$ over all H-free n-vertex 3-uniform hypergraphs G. We determine asymptotically the codegree squared extremal number for various n-vertex 3-uniform hypergraphs including K_4^3 and K_5^3. Further, we prove several general properties about $\text{exco}_2(n,G)$ including the existence of a scaled limit and a supersaturation result.

1 Introduction

For a k-uniform hypergraph H (shortly k-graph), the Turán function (or extremal number) $\text{ex}(n,H)$ is the maximum number of edges in an H-free n-vertex k-uniform hypergraph. The graph case, $k=2$, is reasonably well-understood. The classical Erdős-Stone-Simonovits theorem [14, 16] determines asymptotically the extremal number of graphs with chromatic number at least three. However, for general k, the problem of determining the extremal function is much harder and widely open. Despite enormous efforts, our understanding about Turán functions is still limited. Even the extremal function of the *tetrahedron* K_4^3, the 3-graph on 4 vertices with 4 edges, is unknown. There are exponentially many conjectured extremal examples which is believed to be the root of the hardship of this problem. Brown [10], Kostochka [32], Fon-der-Flaass [21] and Frohmader [23] constructed families of K_4^3-free 3-graphs which they conjectured to be extremal.

Successively, the upper bound for extremal number of the tetrahedron has been improved by de Caen [13], Giraud (unpublished, see [11]), Chung and Lu [11], and finally Razborov [43] and Baber [2] making use of Razborov's flag algebra approach [42] (see also Baber and Talbot [3]).

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Another relevant result towards solving Turán's tetrahedron problem is by Pikhurko [40], building on a result by Razborov [43], who determined the exact extremal hypergraph when additionally the induced 4 vertex graph with one edge is forbidden.

In this paper we will take a different approach on understanding Turán's tetrahedron question, we study a different type of extremality for hypergraphs and solve the tetrahedron problem asymptotically in our notion. Let H be an n-vertex k-uniform hypergraph. For $T \subset V(H)$ with $|T| = k-1$ we denote by $d_H(T)$ the codegree of T, i.e., the number of edges in H containing T. If the choice of H is obvious, we will drop the index and just write $d(T)$. The ℓ_1-norm of the codegrees, or to put it in other words, the sum of codegrees, is k times the number of edges. Thus, Turán's problem for 3-graphs is equivalent to the question of finding the maximum ℓ_1-norm for the codegree vector of G-free graphs. We propose to study the ℓ_2-norm, which we will refer to as the *codegree squared sum* denoted by $\text{co}_2(H)$,

$$\text{co}_2(H) = \sum_{\substack{T \subset \binom{[n]}{k-1} \\ |T|=k-1}} d_H^2(T).$$

Question 1.1. *Given a k-uniform hypergraph G, what is the maximum codegree squared sum a k-uniform G-free n-vertex hypergraph H can have?*

Many different types of extremality in hypergraphs have been studied. The most related one is the minimum codegree-threshold. For a given k-graph, the minimum codegree-threshold is the largest minimum codegree an n-vertex k-graph can have without containing a copy of H. This problem has not even been solved for H being the tetrahedron. For a collection of results on the minimum codegree-threshold see [17–19, 35–39, 46].

In this paper we solve asymptotically Question 1.1 for the tetrahedron. Our result can also be considered as a global stability theorem for the original Turán problem. For a family $\mathcal{F}$ of k-uniform hypergraphs, we define $\text{exco}_2(n, \mathcal{F})$ to be the maximum codegree squared sum a k-uniform n-vertex $\mathcal{F}$-free hypergraph can have, and the *codegree squared density* $\sigma(F)$ to be its scaled limit, i.e.,

$$\text{exco}_2(n, \mathcal{F}) = \max_{\substack{G \text{ is an } n\text{-vertex} \\ \mathcal{F}\text{-free} \\ k\text{-uniform hypergraph}}} \text{co}_2(G) \qquad \text{and} \qquad \sigma(\mathcal{F}) = \limsup_{n\to\infty} \frac{\text{exco}_2(n, \mathcal{F})}{\binom{n}{k-1}(n-k+1)^2}. \quad (1)$$

Denote K_ℓ^3 the complete 3-uniform hypergraph on ℓ vertices. Our main result is that we determine the codegree squared density asymptotically for K_4^3 and K_5^3, respectively.

Theorem 1.2. *We have*

$$\sigma(K_4^3) = \frac{1}{3} \qquad \textit{and} \qquad \sigma(K_5^3) = \frac{5}{8}.$$

Denote C_n the 3-uniform hypergraph on n vertices with vertex set $V(C_n) = V_1 \cup V_2 \cup V_3$ such that $||V_i| - |V_j|| \leq 1$ for $i \neq j$ and edge set

$$\begin{aligned} E(C_n) = \{abc : a \in V_1, b \in V_2, c \in V_3\} &\cup \{abc : a, b \in V_1, c \in V_2\} \\ \cup \{abc : a, b \in V_2, c \in V_3\} &\cup \{abc : a, b \in V_3, c \in V_1\}. \end{aligned}$$

Further, denote B_n the balanced, complete, bipartite 3-uniform hypergraph on n vertices, that is the hypergraph where the vertex set is partitioned $V(G) = A \cup B$ such that $||A| - |B|| \leq 1$ and the edge set is the set of triples intersecting both A and B, see Figure 1 for an illustration of C_n and B_n. We conjecture that C_n and B_n are the corresponding extremal hypergraphs for K_4^3 and K_5^3 respectively.

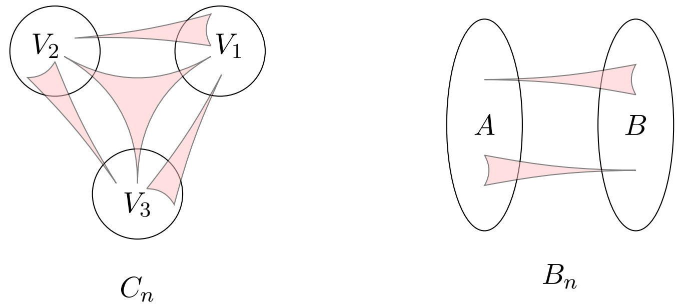


Figure 1: Illustration of C_n and B_n.

Conjecture 1.3. *There exists n_0 such that for all $n \geq n_0$*

$$\mathrm{exco}_2(n, K_4^3) = \mathrm{co}_2(C_n)$$

and C_n is the unique K_4^3-free n-vertex 3-uniform hypergraph with $\mathrm{exco}_2(n, K_4^3)$ edges.

Note that Kostochka's [32] result suggests that in the ℓ_1-norm there are exponentially many extremal graphs, C_n is one of them.

Conjecture 1.4. *There exists n_0 such that for all $n \geq n_0$*

$$\mathrm{exco}_2(n, K_5^3) = \mathrm{co}_2(B_n)$$

and B_n is the unique K_5^3-free n-vertex 3-uniform hypergraph with $\mathrm{exco}_2(n, K_5^3)$ edges.

We believe that existing methods could prove these conjectures, though the potential proofs seem to be long and technical. To motivate our conjectures, we will prove the corresponding stability results. For $\varepsilon > 0$, we say a given n-vertex 3-graph H is *ε-near* to an n-vertex 3-graph G if there exists a bijection $\phi : V(G) \to V(H)$ such that the number of 3-sets $\{x, y, z\}$ satisfying $xyz \in E(G), \phi(x)\phi(y)\phi(z) \notin E(H)$ or $xyz \notin E(G), \phi(x)\phi(y)\phi(z) \in E(H)$ is at most $\varepsilon|V(H)|^3$.

Theorem 1.5. *For every $\varepsilon > 0$ there exists $\delta > 0$ and n_0 such that for every $n > n_0$, if H is a K_4^3-free 3-uniform hypergraph on n vertices with*

$$\mathrm{co}_2(G) \geq \left(\frac{1}{3} - \delta\right)\frac{n^4}{2},$$

then H is ε-near to C_n.

Theorem 1.6. *For every $\varepsilon > 0$ there exists $\delta > 0$ and n_0 such that for every $n > n_0$, if H is a K_5^3-free 3-uniform hypergraph on n vertices with*

$$\mathrm{co}_2(G) \geq \left(\frac{5}{8} - \delta\right)\frac{n^4}{2},$$

then H is ε-near to B_n.

Remark that the two stability results are stronger than Theorem 1.2. Note that C_n is also conjectured to be one of the asymptotically extremal examples for K_4^3 in ℓ_1-norm. However, for K_5^3, B_n is not the Turán extremal example as there is a K_5^3-free 3-graph [45] with higher edge density, namely H_5. The vertex set of H_5 is divided into 4 parts A_1, A_2, A_3, A_4 with $||A_j| - |A_i|| \leq 1$ for all $1 \leq i \leq j \leq 4$ and say a triple e is not an edge of H_5 iff there is

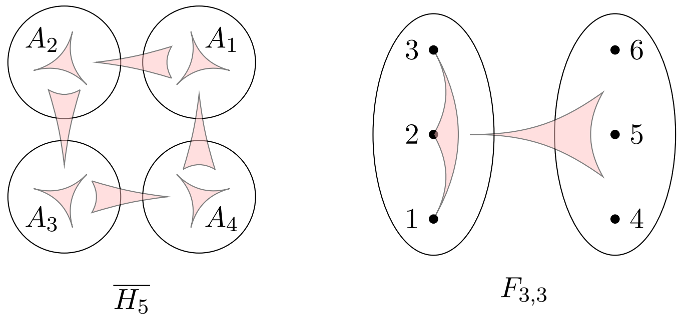


Figure 2: Left: The complement of H_5. Right: A sketch of $F_{3,3}$, which has 6 vertices and edge set $\{123, 145, 146, 156, 245, 246, 256, 345, 346, 356\}$.

some j ($1 \leq j \leq 4$) such that $|e \cap A_j| \geq 2$ and $|e \cap A_j| + |e \cap A_{j+1}| = 3$, where $A_5 = A_1$, see Figure 2 for an illustration of the complement of H_5. Therefore, K_5^3 is an example of a 3-graph where the codegree squared extremal example differs from the Turán extremal example (even asymptotically).

Besides giving asymptotic result for cliques, we prove an exact result for $F_{3,3}$. Denote $F_{3,3}$ the 3-graph on 6 vertices with edge set $\{123, 145, 146, 156, 245, 246, 256, 345, 346, 356\}$, see Figure 2. We prove that the codegree squared extremal example of $F_{3,3}$ is the balanced, complete, bipartite hypergraph B_n. Keevash and Mubayi [30] proved that B_n is extremal also for ℓ_1-norm.

Theorem 1.7. *There exists n_0 such that for all $n \geq n_0$*

$$\mathrm{exco}_2(n, F_{3,3}) = \mathrm{co}_2(B_n).$$

Furthermore, B_n is the unique $F_{3,3}$-free 3-uniform hypergraph H on n vertices satisfying

$$\mathrm{co}_2(H) = \mathrm{exco}_2(n, F_{3,3}).$$

We also prove some general results on σ. First, we prove that the limit in (1) exists.

Proposition 1.8. *Let $\mathcal{F}$ be a family of k-graphs. Then, $\frac{\mathrm{exco}_2(n,\mathcal{F})}{\binom{n}{k-1}(n-k+1)^2}$ is non-increasing as n increases. In particular, its limit exists.*

A classical result in extremal combinatorics is the supersaturation phenomenon, discovered by Erdős and Simonovits [15]. For hypergraphs it states, that when the edge density of a hypergraph H exceeds the Turán density of a different hypergraph G, then H contains many copies of G. Proposition 1.9 shows that the same phenomenon holds for σ.

Proposition 1.9. *Let F be a k-graph on f vertices. For every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, f) > 0$ and n_0 such that every n-vertex k-uniform hypergraph G with $n > n_0$ and $\mathrm{co}_2(G) > (\sigma(F) + \varepsilon)\binom{n}{k-1}n^2$ contains at least $\delta\binom{n}{f}$ copies of F.*

Supersaturation has been used to show that blowing up a k-graph does not change its Turán density [15]. We will use our Supersaturation result, Proposition 1.9, to make the same conclusion for σ: Blowing up a k-graph does also not change its codegree squared density.

For a k-graph H and $t \in \mathbb{N}$, the *blow-up* $H(t)$ of H is defined by replacing each vertex $x \in V(H)$ by t vertices $x^1, \dots, x^t$ and each edge $x_1 \cdots x_k \in E(H)$ by t^k edges $x_1^{a_1} \cdots x_k^{a_k}$ with $1 \leq a_1, \dots, a_k \leq t$.

Corollary 1.10. *Let H be a k-uniform hypergraph and $t \in \mathbb{N}$. Then,*

$$\sigma(H) = \sigma(H(t)).$$

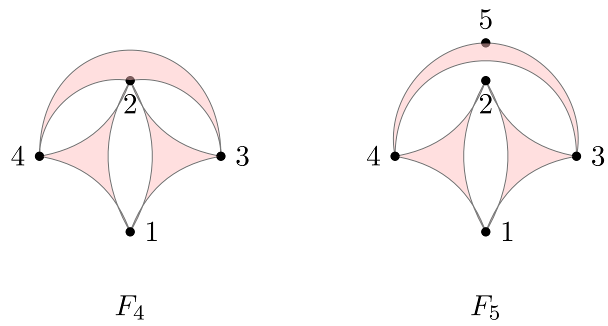


Figure 3: Hypergraphs F_4 and F_5.

Our paper is organized as follows. In Section 2 we calculate the ℓ_2-norm for a classical, but easy, example in ℓ_1-norm as a warm up. Next, in Section 3 we introduce terminology and give an overview on the tools we will be using. In Section 4 we present our general results on codegree squared extremal number. Section 5 is dedicated to proving our main result on cliques, meaning proving Theorems 1.5 and 1.6. In Section 6 we present the proof of our exact result, Theorem 1.7.
In a follow-up paper [4] we shall have a systematic study of codegree squared densities of several hypergraphs, we defer further discussion on open problems there.

2 A Toy Example: Forbidding F_4 and F_5

In this section we will provide an example of how a classical Turán result, ℓ_1-norm, can imply a result for the codegree squared density, ℓ_2-norm. Denote F_4 [1] the 4-vertex 3-graph with edge set $\{123, 124, 234\}$ and F_5 the 5-vertex 3-graph with edge set $\{123, 124, 345\}$, see Figure 3. The 3-graphs which are F_4- and F_5-free are called *cancellative hypergraphs*. Denote by S_n the complete 3-partite 3-graph on n vertices with part sizes $\lfloor n/3\rfloor$, $\lfloor (n+1)/3\rfloor$, $\lfloor (n+2)/3\rfloor$. Bollobás [8] proved that the n-vertex cancellative hypergraph with the most edges is S_n. Using his result and a double counting argument we show that S_n is also the largest cancellative hypergraph in the ℓ_2-norm.

Theorem 2.1. *We have*

$$\mathrm{exco}_2(n, \{F_4, F_5\}) = \mathrm{co}_2(S_n),$$

and therefore also

$$\sigma(\{F_4, F_5\}) = \frac{2}{27}.$$

The unique extremal hypergraph is S_n.

Proof. Let H be an F_4- and F_5-free hypergraph with n vertices. For an edge $e = \{x, y, z\} \in E(H)$, we define the weight $w(e) = d(x, y) + d(x, z) + d(y, z)$. Then, $w(e) \le n$; otherwise H contains an F_4. Bollobás [8] proved that $|E(H)| \le |E(S_n)|$ with equality iff $H = S_n$. This allows us to conclude

$$\mathrm{co}_2(H) = \sum_{xy\in\binom{[n]}{2}} d(x,y)^2 = \sum_{e\in E(H)} w(e) \le n|E(H)| \le n|E(S_n)| = \mathrm{co}_2(S_n),$$

[1] This hypergraph is also knows as K_4^{3-}.

where the second equality holds by realizing that in both expressions $d(x, y)$ is counted $d(x, y)$ times for all $x, y \in \binom{V(G)}{2}$. ■

Frankl and Füredi [22] proved that for just F_5-free 3-graphs, S_n is also the extremal example in ℓ_1-norm when $n \geq 3000$. In a follow-up paper we shall prove that for F_5-free 3-graphs, S_n is also the extremal example in the ℓ_2-norm providing n is sufficiently large. However, this requires more work than the proof of Theorem 2.1 and it is not derived by just applying the corresponding Turán result.

3 Preparation

3.1 Terminology and notation

Let H be a 3-uniform hypergraph, $x \in V(H)$ and $A, B \subseteq V(H)$ be disjoint sets.

1. $L(x)$ denotes the link graph of x, i.e., the graph on $V(H) \setminus \{x\}$ with $ab \in E(L(x))$ iff $abx \in E(H)$.
2. $L_A(x) = L(x)[A]$ denotes the induced link graph on A.
3. $L_{A,B}(x)$ denotes the subgraph of the link graph of x containing only edges between A and B. This means $V(L_{A,B}(x)) = V(H) \setminus \{x\}$ and $ab \in E(L_{A,B}(x))$ iff $a \in A, b \in B$ and $abx \in E(H)$.
4. $L^c_{A,B}(x)$ denotes the subgraph of the link graph of x containing only non-edges between A and B. This means $V(L_{A,B}(x)) = V(H) \setminus \{x\}$ and $ab \in E(L^c_{A,B}(x))$ iff $a \in A, b \in B$ and $abx \notin E(H)$.
5. $e(A, B)$ denotes the number of cross edges between A and B, this means $e(A, B) := |\{xyz \in E(H) : x, y \in A, z \in B\}| + |\{xyz \in E(H) : x, y \in B, z \in A\}|$.
6. $e^c(A, B)$ denotes the number of missing cross edges between A and B, this means $e^c(A, B) := \binom{|A|}{2}|B| + \binom{|B|}{2}|A| - e(A, B)$.
7. For an edge $e = \{x, y, z\} \in E(H)$, we define the weight

$$w_H(e) = d(x, y) + d(x, z) + d(y, z).$$

3.2 Tool 1: Induced hypergraph removal Lemma

We will use the induced hypergraph removal lemma of Rödl and Schacht [44].

Definition 3.1. Let $\mathcal{F}, \mathcal{P}$ be families of k-graphs.

- $\text{Forb}_{ind}(\mathcal{F})$ denotes the family of all k-graphs H which contain no induced copy of any member of $\mathcal{F}$.
- For a constant $\mu \geq 0$ we say a given k-graph H is *μ-far* from $\mathcal{P}$ if every k-graph G on the same vertex set $V(H)$ with $|G \triangle H| \leq \mu |V(H)|^k$ satisfies $G \notin \mathcal{P}$, where $G \triangle H$ denotes the symmetric difference of the edge sets of G and H. Otherwise we call H *μ-near* to $\mathcal{P}$.

Theorem 3.2 (Rödl, Schacht [44])**.** *For every (possibly infinite) family $\mathcal{F}$ of k-graphs and every $\mu > 0$ there exist constants $c > 0, C > 0$, and $n_0 \in \mathbb{N}$ such that the following holds. Suppose H is a k-graph on $n \geq n_0$ vertices. If for every $\ell = 1, \ldots, C$ and every $F \in \mathcal{F}$ on ℓ vertices, H contains at most cn^ℓ induced copies of F, then H is μ-near to $Forb_{ind}(F)$.*

3.3 Tool 2: Flag Algebras

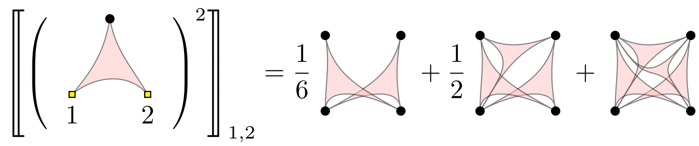
In this section we give an insight on how we apply Razborov's flag algebra machinery [42] for calculating the codegree squared density. The main power comes from the possibility to formulate a problem as a semidefinite program and use a computer to solve it.

The method can be applied in various settings such as graphs [25, 41], hyperrgaphs [3, 18], oriented graphs [26,34], edge-colored graphs [5,12], permutations [6,47], discrete geometry [7,33], or phylogenetic trees [1]. For a good explanations of the flag algebra method in the setting of 3-uniform hypergraphs see [20]. Here, we will focus on the problem formulation rather than a formal explanation of the general method.

Let F be a fixed 3-graph. Let $\mathcal{F}$ denote the set of all F-free 3-graphs up to isomorphism. Denote by $\mathcal{F}_\ell$ all 3-graphs in $\mathcal{F}$ on ℓ vertices. For two 3-graphs F_1 and F_2, denote by $P(F_1, F_2)$ the probability that $|V(F_1)|$ vertices chosen uniformly at random from $V(F_2)$ induce a copy of F_1. A sequence of 3-graphs $(G_n)_{n\geq 1}$ of increasing orders is *convergent*, if $\lim_{n\to\infty} P(H, G_n)$ exists for every $H \in \mathcal{F}$. Notice that if this limit exists, it is in $[0, 1]$.

For readers familiar with flag algebras and its usual notation, for a convergent sequence $(G_n)_{n\geq 1}$ with G_n being n-vertex 3-graphs, we get

$$\lim_{n\to\infty} \frac{\mathrm{co}_2(G_n)}{\binom{n}{2}(n-2)^2} =$$



(2)

where the terms on the right are interpreted as

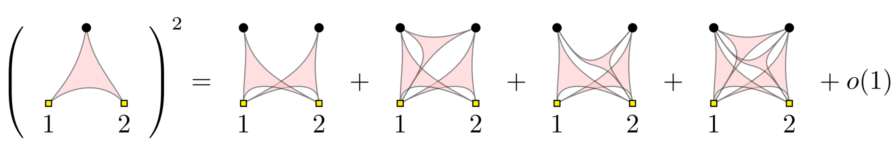
$$\lim_{n\to\infty} \frac{1}{6} P(K_4^{=}, G_n) + \frac{1}{3} P(K_4^{3-}, G_n) + P(K_4^3, G_n),$$

where $K_4^{=}$ is a 3-graph with 4 vertices and 2 edges and K_4^{-} a 3-graph with 4 vertices and 3 edges, also known as F_4. It is a routine application of flag algebras to find an upper bound on the right-hand side of (2).

For readers less familiar with flag algebras, the following paragraphs give a slightly less formal explanation of the problem formulation. Let G be a 3-graph. Let θ be an injective function $\{1, 2\} \to V(G)$. In other words, θ labels two distinct vertices in G. We call the pair (G, θ) a *labeled 3-graph* although only two vertices in G are labeled by θ.

Let (H, θ') and (G, θ) be two labeled 3-graphs. Let X be a subset of $V(G) \setminus \operatorname{Im}\theta$ of size $|V(H)| - 2$ chosen uniformly at random. By $P((H, \theta'), (G, \theta))$ we denote the probability that the labeled subgraph of G induced by X and the two labeled vertices, i.e., $(G[X \cup \operatorname{Im}\theta], \theta)$, is isomorphic to (H, θ'), where the isomorphism maps $\theta(i)$ to $\theta'(i)$ for $i \in \{1, 2\}$.

Let E be a labeled 3-graph consisting of three vertices, two of them labeled, and one edge containing all three of them. Notice that $P(E, (G, \theta))(n - 2)$ is the codegree of $\theta(1)$ and $\theta(2)$ in G. The codegree of $\theta(1)$ and $\theta(2)$ squared is $(P(E, (G, \theta))(n - 2))^2$. One of the tricks in flag algebras is that calculating $P(E, (G, \theta))^2$ in G of order n can be done with error $O(1/n)$ by selecting two distinct vertices in addition to $\theta(1)$ and $\theta(2)$ and examining subgraphs on four vertices instead. In our case, it looks like the following, where $P(H, (G, \theta))$ is depicted simply as H.



(3)

The next step is to sum over all possible choices for θ, there are $n(n-1)$ of them, and divide by 2 since the codegree squared sum is over unordered pairs of vertices, unlike θ. When summing over all possible θ, one could look at all subsets of vertices of size 4 of G and see what the probability is that randomly labeling two vertices among these four by θ gives a labeled 3-graph in (3). These are the coefficients on the right-hand side of (2).

We use flag algebras for Lemmas 5.1, 6.1, and 5.6. The calculations are computer assisted. We use CSDP [9] for numerical solutions of semidefinite programs and SageMath [48] for rounding the numerical solutions to exact ones. The necessary files to perform the calculations that we developed are available at `http://lidicky.name/pub/co2/`.

4 General results: Proofs of Propositions 1.8, 1.9 and 1.10

4.1 The limit exists

Proof of Proposition 1.8. Let $n > m$ be positive integers and let H be an $\mathcal{F}$-free k-graph on vertex set $[n]$ satisfying $\mathrm{co}_2(H) = \mathrm{exco}_2(n, \mathcal{F})$. Take S to be a randomly chosen m-subset of $V(H)$. Now, we calculate the expectation of $\mathrm{co}_2(H[S])$,

$$\begin{aligned}\mathbb{E}[\mathrm{co}_2(H[S])] &= \sum_{T\in\binom{[n]}{k-1}} \mathbb{E}[\mathbf{1}_{\{T\subset S\}} d^2_{H[S]}(T)] = \sum_{T\in\binom{[n]}{k-1}} \mathbb{P}(T\subset S)\mathbb{E}[d^2_{H[S]}(T)|T\subset S] \\ &= \sum_{T\in\binom{[n]}{k-1}} \frac{\binom{m}{k-1}}{\binom{n}{k-1}}\mathbb{E}[d^2_{H[S]}(T)|T\subset S] \geq \sum_{T\in\binom{[n]}{k-1}} \frac{\binom{m}{k-1}}{\binom{n}{k-1}}\mathbb{E}[d_{H[S]}(T)|T\subset S]^2 \\ &= \sum_{T\in\binom{[n]}{k-1}} \frac{\binom{m}{k-1}}{\binom{n}{k-1}}\left(d_H(T)\frac{m-k+1}{n-k+1}\right)^2 = \frac{\binom{m}{k-1}}{\binom{n}{k-1}}\left(\frac{m-k+1}{n-k+1}\right)^2 \mathrm{co}_2(H).\end{aligned}$$

We used that $d_{H[S]}(T)$ conditioned on $T\subset S$ has hypergeometric distribution. There has to exist an m-vertex subset $S'\subset V(H)$ with $\mathrm{co}_2(H[S'])\geq \mathbb{E}[\mathrm{co}_2(H[S])]$. Thus, we conclude that $G := H[S']$ is an m-vertex k-graph satisfying

$$\mathrm{co}_2(G) \geq \frac{\binom{m}{k-1}}{\binom{n}{k-1}}\left(\frac{m-k+1}{n-k+1}\right)^2 \mathrm{co}_2(H).$$

Therefore, since G is $\mathcal{F}$-free,

$$\frac{\mathrm{exco}_2(m,\mathcal{F})}{\binom{m}{k-1}(m-k+1)^2} \geq \frac{\mathrm{co}_2(G)}{\binom{m}{k-1}(m-k+1)^2} \geq \frac{\mathrm{co}_2(H)}{\binom{n}{k-1}(n-k+1)^2} = \frac{\mathrm{exco}_2(n,\mathcal{F})}{\binom{n}{k-1}(n-k+1)^2}.$$

■

4.2 Supersaturation

In this section we prove Proposition 1.9. We will make use of the following tail bound on the hypergeometric distribution.

Lemma 4.1 (e.g. [27] p.29)**.** *Let $\beta,\lambda > 0$ with $\beta+\lambda < 1$. Suppose that $X\subseteq [n]$ and $|X|\geq (\beta+\lambda)n$. Then*

$$\left|\left\{S\in\binom{[n]}{m} : |S\cap X|\leq \beta m\right\}\right| \leq \binom{n}{m}e^{-\frac{\lambda^2 m}{3(\beta+\lambda)}} \leq \binom{n}{m}e^{-\lambda^2 m/3}.$$

Mubayi and Zhao [38] used Lemma 4.1 to prove a supersaturation result for the minimum codegree threshold. We adjust their proof to our setting.

Lemma 4.2. *Given $\alpha, \varepsilon > 0, k \geq 3$. Then there exists m_0 such that the following holds. If $n \geq m \geq m_0$ and G is a k-graph on $[n]$ with $\mathrm{co}_2(G) \geq (\alpha+\varepsilon)\binom{n}{k-1}(n-k+1)^2$, then the number of m-sets S satisfying $\mathrm{co}_2(G[S]) > \alpha\binom{m}{k-1}(m-k+1)^2$ is at least $\frac{\varepsilon}{4}\binom{n}{m}$.*

Proof. Given a $(k-1)$-element set $T \subset [n]$, we call an m-set S with $T \subset S \subset [n]$ *bad for T* if $|d(T) \cap S| \leq \left(\frac{d(T)}{n-k+1} - \frac{\varepsilon}{6}\right)(m-k+1)$. An m-set is *bad* if it is bad for some T. Otherwise, it is *good*. We will show that there are few bad sets. Denote Φ the number of bad m-sets, and let Φ_T be the number of m-sets that are bad for T. Then, by applying Lemma 4.1 with $\beta = \frac{d(T)}{n-k+1} - \frac{\varepsilon}{6}$ and $\lambda = \varepsilon/7$, we get

$$\begin{aligned}\Phi &\leq \sum_{T \in \binom{[n]}{k-1}} \Phi_T = \sum_{T \in \binom{[n]}{k-1}} \left|\left\{S' \in \binom{[n]\setminus T}{m-k+1} : |d(T) \cap S'| \leq \left(\frac{d(T)}{n-k+1} - \frac{\varepsilon}{6}\right)(m-k+1)\right\}\right| \\ &\leq \sum_{T \in \binom{[n]}{k-1}} \binom{n-k+1}{m-k+1} \exp\left(-\frac{\varepsilon^2(m-k+1)}{147}\right) \leq \binom{n}{k-1}\binom{n-k+1}{m-k+1} \exp\left(-\frac{\varepsilon^2(m-k+1)}{147}\right) \\ &= \binom{n}{m}\binom{m}{k-1} \exp\left(-\frac{\varepsilon^2(m-k+1)}{147}\right) \leq \frac{\varepsilon}{4}\binom{n}{m},\end{aligned}$$

where the last inequality holds for m large enough. So the number of bad m-sets is at most $\frac{\varepsilon}{4}\binom{n}{m}$. Now let $\ell\binom{n}{m}$ be the number of m-sets S satisfying

$$\sum_{T \in \binom{S}{k-1}} d_G^2(T) \geq \left(\alpha + \frac{\varepsilon}{2}\right)\binom{m}{k-1}(n-k+1)^2. \tag{4}$$

On one side

$$\sum_{|S|=m} \sum_{T \in \binom{S}{k-1}} d_G^2(T) = \binom{n-k+1}{m-k+1}\mathrm{co}_2(G) = \binom{n-k+1}{m-k+1}\binom{n}{k-1}(n-k+1)^2(\alpha+\varepsilon).$$

On the other side,

$$\begin{aligned}\sum_{|S|=m} \sum_{T \in \binom{S}{k-1}} d_G^2(T) &\leq (\alpha + \frac{\varepsilon}{2})\binom{m}{k-1}(n-k+1)^2\binom{n}{m} + \ell\binom{m}{k-1}(n-k+1)^2\binom{n}{m} \\ &= (\alpha + \frac{\varepsilon}{2} + \ell)\binom{m}{k-1}(n-k+1)^2\binom{n}{m}.\end{aligned}$$

By this double counting argument, we conclude $\ell \geq \varepsilon/2$. Since the number of bad m-sets is at most $\frac{\varepsilon}{4}\binom{n}{m}$, there are at least $\frac{\varepsilon}{4}\binom{n}{m}$ good m-sets satisfying (4). All of these m-sets satisfy

$$\begin{aligned}
\mathrm{co}_2(G[S]) &= \sum_{T\in\binom{S}{k-1}} d^2_{G[S]}(T) \geq \sum_{T\in\binom{S}{k-1}} \left(\left(\frac{d_G(T)}{n-k+1}-\frac{\varepsilon}{6}\right)(m-k+1)\right)^2 \\
&= \frac{(m-k+1)^2}{(n-k+1)^2} \sum_{T\in\binom{S}{k-1}} \left(d_G(T)-\frac{\varepsilon}{6}(n-k+1)\right)^2 \\
&\geq \frac{(m-k+1)^2}{(n-k+1)^2} \sum_{T\in\binom{S}{k-1}} \left(d^2_G(T)-\frac{\varepsilon}{3}(n-k+1)^2\right) \\
&\geq \frac{(m-k+1)^2}{(n-k+1)^2} \left(\left(\alpha+\frac{\varepsilon}{2}\right)\binom{m}{k-1}(n-k+1)^2-\binom{m}{k-1}\frac{\varepsilon}{3}(n-k+1)^2\right) \\
&> \alpha\binom{m}{k-1}(m-k+1)^2,
\end{aligned}$$

proving the statement of this lemma. ■

Proof of Proposition 1.9. This proof follows Erdős' and Simonovits' proof [15] of the supersaturation result for the Turán density.

Let F be a k-graph on f vertices, $\varepsilon > 0$ and G be an n-vertex k-graph satisfying $\mathrm{co}_2(G) > (\sigma(F)+\varepsilon)\binom{n}{k-1}n^2$ for n large enough. By Lemma 4.2, there exists an m_0 such that for $m \geq m_0$ the number of m-sets S satisfying $\mathrm{co}_2(G[S]) > (\sigma(F)+\varepsilon/2)\binom{m}{k-1}(m-k+1)^2$ is at least $\frac{\varepsilon}{8}\binom{n}{m}$. Thus, for m large enough there are at least $\frac{\varepsilon}{8}\binom{n}{m}$ sets S such that $G[S]$ contains F. Each copy of F may be counted at most $\binom{n-f}{m-f}$ times. Therefore, the number of copies for F is at least

$$\frac{\frac{\varepsilon}{8}\binom{n}{m}}{\binom{n-f}{m-f}} = \delta\binom{n}{f},$$

for some $\delta = \delta(\varepsilon, f)$. ■

4.3 Blowing-up does not change the codegree squared density

Now we use a standard argument to show that blowing-up a k-graph does not change the codegree squared density. We will follow the proof of the analog Turán result given in [28].

Proof of Corollary 1.10. Since $H \subset H(t)$, $\mathrm{exco}_2(n, H(t)) \leq \mathrm{exco}_2(n, H)$ holds trivially. Thus, $\sigma(H(t)) \leq \sigma(H)$.
For the other inequality, let $\varepsilon > 0$ and G be an n-vertex k-uniform hypergraph satisfying $\mathrm{co}_2(G)/(\binom{n}{k-1}(n-k+1)^2) > \sigma(H)+\varepsilon$. Then, by Proposition 1.9, G contains at least $\delta\binom{n}{v(H)}$ copies of H for $\delta = \delta(\varepsilon, k) > 0$. We create an auxiliary $v(H)$-graph F on vertex set $V(G)$. A $v(H)$-set $A \subset V(G)$ is an edge in F iff $G[A]$ contains a copy of H. The auxiliary hypergraph F has density at least $\delta/v(H)!$. Thus, for any $t' > 0$ as long as n is large enough, F contains a copy of the complete $v(H)$-partite $v(H)$-graph with t' vertices in each part $K^{v(H)}_{v(H)}(t')$.
We choose t' large enough such that the following is true. We color each edge of $K^{v(H)}_{v(H)}(t')$ by one of $v(H)!$ colors, depending on which of the $v(H)!$ orders the vertices of H are mapped to the corresponding copy of H in G. A classical result in Ramsey theory gives that there is a monochromatic copy of $K^{v(H)}_{v(H)}(t)$, which gives a copy of $H(t)$ in G. We conclude $\sigma(H(t)) \leq \sigma(H)+\varepsilon$ for all $\varepsilon > 0$. ■

5 Cliques

In this section we will prove Theorems 1.5 and 1.6.

5.1 Proof of Theorem 1.5

Flag algebras give us the following results for K_4^3.

Lemma 5.1. *For all $\varepsilon > 0$ there exists $\delta > 0$ and n_0 such that for all $n \geq n_0$: if H is a K_4^3-free 3-uniform graph on n vertices with $\mathrm{co}_2(H) \geq (1-\delta)\frac{1}{3}n^4/2$, then the densities of all 3-graphs on 4, 5 and 6 vertices in H that are not contained in C_n are at most ε. Additionally,*

$$\sigma(K_4^3) = \frac{1}{3}.$$

The flag algebra calculation proving Lemma 5.1 is computer assisted and not practical to fit in the paper. The calculation is available at `http://lidicky.name/pub/co2/`. Next, we prove that this flag algebra result implies the corresponding stability result, Theorem 1.5.

Proof of Theorem 1.5. Let $\varepsilon > 0$. During the proof we will use the following constants:

$$1 \gg \varepsilon \gg \delta_4 \gg \delta_3 \gg \delta_2 \gg \delta_1 \gg \delta \gg 0.$$

The constants are chosen in this order from left to right where each constant is a sufficiently small positive number depending only on the previous ones. By applying Lemma 5.1, we get $\delta = \delta(\delta_1) > 0$ such that for all n large enough: If H is a K_4^3-free 3-uniform hypergraph on n vertices with $\mathrm{co}_2(H) \geq (1-\delta)\frac{1}{6}n^4$, then the densities of all 3-graphs on 4, 5 and 6 vertices in H that are not contained in C_n are at most δ_1. Now, we apply the induced hypergraph removal lemma Theorem 3.2 to obtain H' where H' is δ_2-near to H, and H' contains only those graphs on 4, 5 and 6 subgraphs which have positive density in C_n. We have

$$\mathrm{co}_2(H') \geq \mathrm{co}_2(H') - 6\delta_2 n^4 \geq (1-\delta)\frac{1}{6}n^4 - 6\delta_2 n^4 \geq (1 - 37\delta_2)\frac{1}{6}n^4,$$

because when one edge is removed from a 3-uniform hypergraph, then the codegree squared sum can go down by at most $6n$. Take a, b, c, d such that $abc, abd \in E(H')$, and $acd, bcd \notin E(H')$. Note that such a subgraph exists, because 3-graphs G with the property that every 4-set induces 0, 1 or 3 edges have edge density $o(1)$ and thus also $\mathrm{co}_2(G) = o(n^4)$. Flag algebras can verify this. Define disjoint sets $V_1, V_2, V_3 \subset V(H')$ by

- $V_1 := \{v \in V(H') \setminus \{a,b,c,d\} : vab \in E(H'),\ vac, vad, vbc, vbd, vcd \notin E(H')\}$,
- $V_2 := \{v \in V(H') \setminus \{a,b,c,d\} : vbc, vbd, vac, vad, vcd \in E(H'),\ vab \notin E(H')\}$,
- $V_3 := \{v \in V(H') \setminus \{a,b,c,d\} : vac, vad, vbc, vbd \in E(H'),\ vab, vcd \notin E(H')\}$.

Remark that $V(G) = V_1 \cup V_2 \cup V_3 \cup \{a,b,c,d\}$ as otherwise a, b, c, d, v form a forbidden subgraph for $v \in V(G) \setminus (V_1 \cup V_2 \cup V_3 \cup \{a,b,c,d\})$. For convenience set $V_4 := V_1$. See Figure 4 for an illustration of this partition when applied to C_n.

Claim 5.2. *V_1, V_2 and V_3 are independent sets in H'.*

Proof. Assume there is an edge $w_1w_2w_3 \in E(H'[V_1])$. For $1 \leq i < j \leq 3$, because w_i, w_j, a, b, c, d induce a subgraph being an induced subgraph in C_n,

$$w_iw_ja, w_iw_jb, w_iw_jc, w_iw_jd \notin E(H').$$

Now, $\{w_1, w_2, w_3, c\}$ spans exactly one edge in H'. There is no 4-vertex induced subgraph with exactly one edge in C_n, a contradiction. Thus, V_1 is an independent set.

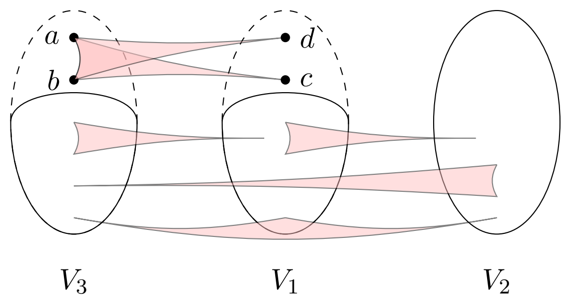


Figure 4: Vertices a, b, c, d and a partition to V_1, V_2, and V_3 in C_n.

Assume there is an edge $w_1w_2w_3 \in E(H'[V_2])$. For $1 \leq i < j \leq 3$, because w_i, w_j, a, b, c, d induce a subgraph being an induced subgraph in C_n,

$$w_iw_ja, w_iw_jb \in E(H') \quad \text{and} \quad w_iw_jc, w_iw_jd \notin E(H').$$

Now, $\{w_1, w_2, w_3, a\}$ spans a K_4^3 in H', a contradiction. Thus, V_2 is an independent set.
Assume there is an edge $w_1w_2w_3 \in E(H'[V_3])$. For $1 \leq i < j \leq 3$, because w_i, w_j, a, b, c, d induce a subgraph being an induced subgraph in C_n,

$$w_iw_ja, w_iw_jb \notin E(H') \quad \text{and} \quad w_iw_jc, w_iw_jd \in E(H').$$

Now, $\{w_1, w_2, w_3, c\}$ spans a K_4^3 in H', a contradiction. Thus, V_3 is an independent set. ■

Claim 5.3. *For $1 \leq i \leq 3$, let $w_1, w_2 \in V_{i+1}, w_3 \in V_i$. Then, $w_1w_2w_3 \notin E(H')$.*

Proof. Assume there is an edge $w_1w_2w_3 \in E(H')$ where $w_1, w_2 \in V_2$ and $w_3 \in V_1$. For $1 \leq i < j \leq 3$, because w_i, w_j, a, b, c, d induce a subgraph being an induced subgraph in C_n,

$$\begin{aligned}
&w_1w_2a, w_1w_2b \in E(H') \quad \text{and} \quad w_1w_2c, w_1w_2d \notin E(H'),\\
&w_1w_3a, w_1w_3b, w_1w_3c, w_1w_3d \in E(H'),\\
&w_2w_3a, w_2w_3b, w_2w_3c, w_2w_3d \in E(H').
\end{aligned}$$

Now, $\{w_1, w_2, w_3, a\}$ spans a K_4^3 in H', a contradiction.
Assume there is an edge $w_1w_2w_3 \in E(H')$ where $w_1, w_2 \in V_3$ and $w_3 \in V_2$. For $1 \leq i < j \leq 3$, because w_i, w_j, a, b, c, d induce a subgraph being an induced subgraph in C_n,

$$\begin{aligned}
&w_1w_2a, w_1w_2b \notin E(H') \quad \text{and} \quad w_1w_2c, w_1w_2d \in E(H'),\\
&w_1w_3a, w_1w_3b \notin E(H') \quad \text{and} \quad w_1w_3c, w_1w_3d \in E(H'),\\
&w_2w_3a, w_2w_3b \notin E(H') \quad \text{and} \quad w_2w_3c, w_2w_3d \in E(H').
\end{aligned}$$

Now, $\{w_1, w_2, w_3, c\}$ spans a K_4^3 in H', a contradiction.
Last, assume there is an edge $w_1w_2w_3 \in E(H')$ where $w_1, w_2 \in V_1$ and $w_3 \in V_3$. For $1 \leq i < j \leq 3$, because w_i, w_j, a, b, c, d induce a subgraph being an induced subgraph in C_n,

$$\begin{aligned}
&w_1w_2a, w_1w_2b, w_1w_2c, w_1w_2d \notin E(H'),\\
&w_1w_3a, w_1w_3b \in E(H') \quad \text{and} \quad w_1w_3c, w_1w_3d \notin E(H'),\\
&w_2w_3a, w_2w_3b \in E(H') \quad \text{and} \quad w_2w_3c, w_2w_3d \notin E(H').
\end{aligned}$$

Now, $\{w_1, w_2, w_3, c\}$ spans exactly one edges in H'. There is no 4-vertex induced subgraph with exactly one edge in C_n, a contradiction. ■

Set H'' to be the 3-graph created from H' by removing all edges incident to a, b, c or d. In total we have removed at most $3n^2$ edges. For each edge, the codegree squared sum can be lowered by at most $6n$. Thus, we have

$$\mathrm{co}_2(H'') \geq \mathrm{co}_2(H') - 18n^3 \geq (1 - 37\delta_2)\frac{1}{6}n^4 - 18n^3 \geq (1 - 40\delta_2)\frac{1}{6}n^4.$$

Further, by Claims 5.3 and 5.2, there is a vertex partition $V(H'') = A \cup B \cup C$, such that A, B and C are independent sets, $|A| = \alpha n, |B| = \beta n, |C| = \gamma n$, and there are no edges $w_1w_2w_3$ with $w_1, w_2 \in A, w_3 \in C$ or $w_1, w_2 \in B, w_3 \in A$ or $w_1, w_2 \in C, w_3 \in B$. Without loss of generality let $\alpha \leq \beta, \alpha \leq \gamma$. Now, we bound the class sizes.

Claim 5.4. *We have* $\alpha, \beta, \gamma \geq \frac{1}{3} - \delta_3$.

Proof. Assume $\alpha \leq \frac{1}{3} - \delta_3$. Then,

$$\begin{aligned}(1 - 40\delta_2)\frac{1}{6}n^4 &\leq \mathrm{co}_2(H'')\\ &\leq \left(\frac{\alpha^2}{2}\beta^2 + \frac{\beta^2}{2}\gamma^2 + \frac{c^2}{2}\alpha^2 + \alpha\beta(\alpha+\gamma)^2 + \beta\gamma(\alpha+\beta)^2 + \gamma\alpha(\gamma+\beta)^2\right)n^4\\ &= \left(\frac{\alpha^2}{2}\beta^2 + \frac{\beta^2}{2}\gamma^2 + \frac{\gamma^2}{2}\alpha^2 + \alpha^3\beta + \beta^3\gamma + \gamma^3\alpha + 3\alpha\beta\gamma\right)n^4 < (1 - 40\delta_2)\frac{1}{6}n^4,\end{aligned}$$

a contradiction. We used that the function $\frac{\alpha^2}{2}\beta^2 + \frac{\beta^2}{2}\gamma^2 + \frac{\gamma^2}{2}\alpha^2 + \alpha^3\beta + \beta^3\gamma + \gamma^3\alpha + 3\alpha\beta\gamma$ is continuous and has a unique maximum of $1/6$ attained at $\alpha = \beta = \gamma = 1/3$. This can be checked using a computer. ■

Next, we can lower bound the number of edges of H''.

Claim 5.5. *We have*

$$|E(H'')| \geq \left(\frac{5}{54} - \delta_4\right)n^3.$$

Proof. Denote E_1 the set of edges in H'' with one vertex from each set A, B, C, and denote E_2 the remaining edges in H''. Then, we have

$$(1 - 40\delta_2)\frac{1}{6}n^4 \leq \mathrm{co}_2(H'') = \sum_{e \in E(H'')} w_{H''}(e) \leq |E_1|(2 + 6\delta_3)n + |E_2|\left(\frac{5}{3} + 6\delta_3\right)n.$$

Since $|E_1| \leq |A||B||C| \leq \frac{1}{27}n^3$, we get

$$|E_2| \geq \frac{(1 - 40\delta_2)\frac{1}{6} - (2 + 6\delta_3)\frac{1}{27}}{\left(\frac{5}{3} + 6\delta_3\right)}n^3 \geq \frac{\frac{5}{54} - \delta_3}{\frac{5}{3} + 6\delta_3}n^3 \geq \left(\frac{1}{18} - \frac{1}{2}\delta_4\right)n^3.$$

Similarly, since

$$|E_2| \leq \frac{3}{2}\left(\frac{1}{3} + 2\delta_3\right)^3 \leq \left(\frac{1}{18} + 2\delta_3\right)n^3,$$

we have $|E_1| \geq \left(\frac{1}{27} - \frac{1}{2}\delta_4\right)n^3$. Thus,

$$|E(H'')| = |E_1| + |E_2| \geq \left(\frac{5}{54} - \delta_4\right)n^3.$$

■

We conclude by Claims 5.4 and 5.5 that H'' is $\varepsilon/2$-near to C_n. Since H'' is $2\delta_2$-near to H, we conclude that H is ε-near to C_n.

■

5.2 Proof of Theorem 1.6

Flag algebras give us the following for K_5^3.

Lemma 5.6. *For all $\varepsilon > 0$ there exists $\delta > 0$ and n_0 such that for all $n \geq n_0$: if H is a K_5^3-free 3-uniform graph on n vertices with $\mathrm{co}_2(H) \geq (1-\delta)\frac{5}{8}n^4/2$, then the densities of all 3-graphs on 4, 5 and 6 vertices in H that are not contained in B_n are at most ε. In particular,*

$$\sigma(K_5^3) = \frac{5}{8}.$$

Again, the flag algebra calculation proving Lemma 5.6 is computer assisted and available at `http://lidicky.name/pub/co2/`. We use this result to prove Theorem 1.6.

Proof of Theorem 1.6. Let $\varepsilon > 0$. During the proof we will use the following constants:

$$1 \gg \varepsilon \gg \delta_2 \gg \delta_1 \gg \delta \gg 0.$$

The constants are chosen in this order and each constant is a sufficiently small positive number depending only on the previous ones. Apply Lemma 5.6 and get $\delta = \delta(\delta_1) > 0$ such that for all n large enough: If H is an K_5^3-free 3-uniform graph on n vertices with $\mathrm{co}_2(H) \geq (1-\delta)\frac{5}{8}n^4/2$, then the densities of all 3-graphs on 4, 5 and 6 vertices in H that are not contained in B_n are at most δ_1.
Now, apply the induced hypergraph removal lemma Theorem 3.2 to obtain H' where H' is δ_2-near to H, and H' contains only those induced subgraphs on 4, 5 or 6 vertices which appear as induced subgraphs in B_n. Note that

$$\mathrm{co}_2(H') \geq \mathrm{co}_2(H') - 6\delta_2 n^4 \geq (1-\delta)\frac{5}{8}\frac{n^4}{2} - 6\delta_2 n^4 \geq (1-20\delta_2)\frac{5}{8}\frac{n^4}{2},$$

because when one edge is removed the codegree squared sum can go down by at most $6n$. Next we show that H' has to have the same structure as B_n. We say that a 3-graph G is 2-*colorable*, if there is a partition of the vertex set $V(G) = V_1 \cup V_2$ such that V_1 and V_2 are independent sets in G.

Claim 5.7. *H' is 2-colorable.*

Proof. Take an arbitrary non-edge abc in H'. For $0 \leq i \leq 4$, define A_i to be the set of vertices $v \in V(G) \setminus \{a,b,c\}$ such that H' induces i edges on $\{a,b,c,v\}$. Then, $A_1 = A_2 = A_4 = \emptyset$ because on 4 vertices there are either 0 or 3 edges in B_n. Clearly, A_0 is an independent set, because if there is an edge $v_1v_2v_3$ in $H'[A_0]$, then the induced graph of H' on a,b,c,v_1,v_2,v_3 forms an forbidden subgraph. Similarly, A_3 is an independent set. Let $A' = A_0 \cup \{a,b,c\}$. Then $V(H') = A_3 \cup A'$ and A' also forms an independent set. To observe the second statement, let v_1, v_2, v_3 be three vertices in A_0. The number of edges induced on v_1, v_2, v_3, a, b, c is at most nine, because every edge needs to be incident to exactly two vertices of $\{a,b,c\}$ by the definition of A_0. However, 6-vertex induced subgraphs of B_n have either 0, 10, 16, or 18 edges. We conclude that v_1, v_2, v_3, a, b, c induce no edges in H'. Thus, A' is also an independent set in H' and therefore H' is 2-colorable. ∎

Claim 5.8. *We have $|E(H')| \geq (1 - 2\sqrt{\delta_2})\frac{n^3}{8}$.*

Proof. By Claim 5.7, H' is 2-colorable and we can partition the vertex set $V(H') = A \cup B$ such that A and B are independent sets, and $|A| = an$ and $|B| = bn$ with $a \leq b$. We have

$$(1-20\delta_2)\,\frac{5}{8}\frac{n^4}{2} \leq \mathrm{co}_2(H') \leq \left(\frac{a^2}{2}b^2 + \frac{b^2}{2}a^2 + ab\right)n^4 \leq (ab(ab+1))\,n^4 \leq \frac{5}{4}abn^4.$$

Thus, $4ab \geq 1 - 20\delta_2$. We conclude $a \geq 1/2 - 3\sqrt{\delta_2}$, otherwise

$$4ab < 4\left(\frac{1}{2} - 3\sqrt{\delta_2}\right)\left(\frac{1}{2} + 3\sqrt{\delta_2}\right) = 1 - 36\delta_2,$$

a contradiction. For every edge $e \in E(H')$, we have $w_{H'}(e) \leq \left(5/2 + 3\sqrt{\delta_2}\right)n$. Therefore,

$$(1 - 20\delta_2)\,\frac{5}{8}\frac{n^4}{2} \leq \mathrm{co}_2(H') \leq \sum_{e \in E(H')} w_{H'}(e) \leq |E(H')|\left(\frac{5}{2} + 3\sqrt{\delta_2}\right)n.$$

Thus, $|E(H')| \geq (1 - 2\sqrt{\delta_2})\frac{n^3}{8}$. ■

The 3-graph H is δ_2-near to H'. By Claims 5.7 and 5.8, H' is $\varepsilon/2$-near to B_n. Therefore we can conclude that H is $\delta_2 + \varepsilon/2 \leq \varepsilon$-near to B_n. ■

5.3 Discussion on Cliques

Let H_ℓ be the following 3-graph on n-vertices: Divide the vertex set of H_ℓ into $\ell - 1$ parts $A_1, \ldots, A_{\ell-1}$ with $||A_j| - |A_i|| \leq 1$ for all $1 \leq i \leq j \leq \ell - 1$ and say a triple e is not an edge of H_ℓ iff there is some j $(1 \leq j \leq \ell - 1)$ such that

$$|e \cap A_j| \geq 2 \qquad \text{and} \qquad |e \cap A_j| + |e \cap A_{j+1}| = 3,$$

where $A_\ell = A_1$. It is conjectured that H_ℓ has maximum number of hyperedges among n-vertex K_ℓ^3-free hypergraphs. Note that $H_4 = C_n$, but $H_5 \neq B_n$. In fact, the edge density of B_n is less than H_5. As it was pointed out in the introduction, this means that the asymptotical extremal example for K_5^3 is different in ℓ_1- and ℓ_2-norm. We conjecture that the extremal examples are different for all cliques of order at least 5.
For ℓ odd, denote G_ℓ the following n-vertex 3-graph. Divide the vertex set of G_ℓ into $k = (\ell-1)/2$ parts $A_1, \ldots, A_k$ with $||A_j| - |A_i|| \leq 1$ for all $1 \leq i \leq j \leq k$ and say a set B of size 3 is an edge of G_ℓ if B intersects at least 2 parts; see Figure 5. The hypergraph G_ℓ is K_ℓ^3-free because for every choice of ℓ vertices there are always three of them lying inside the same class.

Question 5.9. *Let $\ell \geq 7$ odd and $k = (\ell - 1)/2$. Is*

$$\sigma(K_\ell^3) = \lim_{n\to\infty} \frac{\mathrm{co}_2(G_\ell)}{\binom{n}{2}(n-2)^2} = 2\left(1 - \frac{1}{k}\right)\left(\frac{1}{2k} - \frac{1}{2k^2} + \frac{1}{2}\right)\ ?$$

For ℓ even, $k = (\ell-4)/2$ and $0 \leq b \leq n/k$ denote $G_{\ell,b}$ the following n-vertex 3-graph. Divide the vertex set of $G_{\ell,b}$ into blocks $B_1, B_2, \ldots B_k, C_1, C_2, C_3$ where $b = |B_1| = |B_2| = \cdots = |B_k|$ and $|C_1| = |C_2| = |C_3| = c$ such that $3c + bk = n$. The edge set induced on $C_1 \cup C_2 \cup C_3$ forms a C_{3c}. Further, all triples intersecting in exactly two or one vertex in one of the blobs B_i's are edges; see Figure 5. Among all 3-graphs $G_{k,b}$, let G_ℓ be the one optimizing the codegree squared sum.

Question 5.10. *Let $\ell \geq 6$ even. Is*

$$\sigma(K_\ell^3) = \lim_{n\to\infty} \frac{\mathrm{co}_2(G_\ell)}{\binom{n}{2}(n-2)^2}\ ?$$

6 Proof of Theorem 1.7

In this section we prove Theorem 1.7, i.e., we determine the codegree squared extremal number of $F_{3,3}$. Flag algebras give us the following corresponding asymptotical result and also a weak stability version.

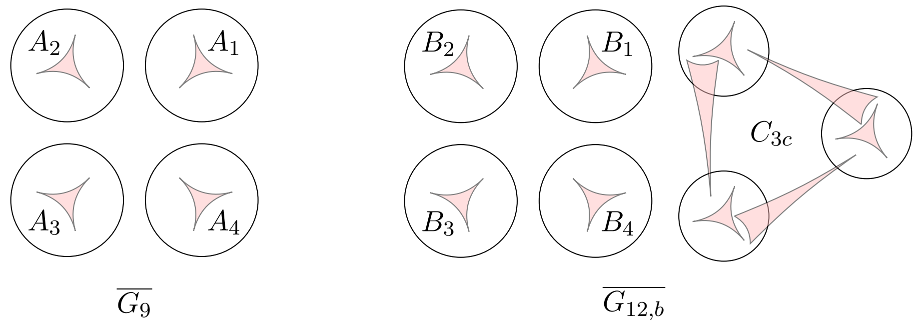


Figure 5: Hypergraph $\overline{G_9}$ does not contain $\overline{K_9^3}$ and $\overline{G_{12,b}}$ does not contain $\overline{K_{12}^3}$.

Lemma 6.1. *For all $\varepsilon > 0$ there exists $\delta > 0$ and n_0 such that for all $n \geq n_0$: if H is an $F_{3,3}$-free 3-uniform graph on n vertices with $\mathrm{co}_2(H) \geq (1-\delta)\frac{5}{8}n^4/2$, then the densities of all 3-graphs on 4, 5 and 6 vertices in H that are not contained in B_n are at most ε. Additionally,*

$$\sigma(F_{3,3}) = \frac{5}{8}.$$

This result implies the following stability theorem.

Theorem 6.2. *For every $\varepsilon > 0$ there is $\delta > 0$ and n_0 such that if H is an $F_{3,3}$-free 3-uniform hypergraph on $n \geq n_0$ vertices with $\mathrm{co}_2(H) \geq (1-\delta)\frac{5}{8}\frac{n^4}{2}$, then we can partition $V(H) = A \cup B$ such that $e(A) + e(B) \leq \varepsilon n^3$ and $e(A,B) \geq \frac{1}{8}n^3 - \varepsilon n^3$.*

Proof. The proof is the same as the proof of Theorem 1.6, except instead of applying Lemma 5.6 we apply Lemma 6.1. ■

Furthermore, we determine the exact extremal number by using the stability result, Theorem 6.2, and a standard cleaning technique, see for example [24,29,31,40]. To do so we will first prove the statement under an additional universal minimum-degree-type assumption.

Theorem 6.3. *There exists n_0 such that for all $n \geq n_0$ the following holds. Let H be an $F_{3,3}$-free n-vertex 3-graph such that*

$$q(x) := \sum_{y \in V, y \neq x} d(x,y)^2 + 2 \sum_{vw \in L(x)} d(x,y) \geq \frac{5}{4}n^3 - 6n^2 =: d(n) \tag{5}$$

for all $x \in V(G)$. Then,

$$\mathrm{co}_2(H) \leq \mathrm{co}_2(B_n) = \binom{\lceil \frac{n}{2} \rceil}{2} \left\lfloor \frac{n}{2} \right\rfloor^2 + \binom{\lfloor \frac{n}{2} \rfloor}{2} \left\lceil \frac{n}{2} \right\rceil^2 + \left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor (n-2)^2.$$

Furthermore, B_n is the unique such 3-graph H satisfying $\mathrm{co}_2(H) = \mathrm{exco}_2(n, F_{3,3})$.

Proof. Let H be a 3-uniform $F_{3,3}$-free hypergraph which has a codegree squared sum of at least $\mathrm{co}_2(H) \geq \mathrm{co}_2(B_n)$ and satisfies (5). Choose $\varepsilon = 10^{-10}$ and apply Theorem 6.2. We get a vertex partition $A \cup B$ with $e(A) + e(B) \leq \varepsilon n^3$ and $e^c(A,B) \leq \varepsilon n^3$. Among all such partitions choose one which minimizes $e(A) + e(B)$. We can assume that $|L_B(x)| \geq |L_A(x)|$ for all $x \in A$ and $|L_A(x)| \geq |L_B(x)|$ for all $x \in B$, as otherwise we could put this vertex to the other class and decrease $e(A) + e(B)$. This is not possible, because we chose A and B minimizing $e(A) + e(B)$. We start by making an observation about the class sizes.

Claim 6.4. *We have*

$$\left||A| - \frac{n}{2}\right| \le 2\sqrt{\varepsilon}n \quad \textit{and} \quad \left||B| - \frac{n}{2}\right| \le 2\sqrt{\varepsilon}n.$$

Proof. Assume that $|A| < n/2 - 2\sqrt{\varepsilon}n$. Then, we have

$$\begin{aligned} e(A,B) &\le \binom{|A|}{2}|B| + |A|\binom{|B|}{2} \le \frac{1}{2}|A|(n-|A|)n \\ &< \frac{1}{2}\left(\frac{n}{2} - 2\sqrt{\varepsilon}n\right)\left(\frac{n}{2} + 2\sqrt{\varepsilon}n\right)n < \frac{1}{8}n^3 - \varepsilon n^3, \end{aligned}$$

a contradiction. Thus, $|A| \ge n/2 - 2\sqrt{\varepsilon}n$. Similarly, we get $|B| \ge n/2 - 2\sqrt{\varepsilon}n$. ∎

Define *junk* sets J_A, J_B to be the sets of vertices which are not typical, i.e.,

$$\begin{aligned} J_A &:= \left\{x \in A : |L^c_{A,B}(x)| \ge \sqrt{\varepsilon}n^2\right\} \cup \left\{x \in A : |L_A(x)| \ge \sqrt{\varepsilon}n^2\right\}, \text{ and} \\ J_B &:= \left\{x \in B : |L^c_{A,B}(x)| \ge \sqrt{\varepsilon}n^2\right\} \cup \left\{x \in B : |L_B(x)| \ge \sqrt{\varepsilon}n^2\right\}. \end{aligned}$$

These junk sets need to be small.

Claim 6.5. *We have $|J_A|, |J_B| \le 5\sqrt{\varepsilon}n$.*

Proof. Towards contradiction assume that $|J_A| > 5\sqrt{\varepsilon}n$. Then the number of vertices $x \in J_A$ satisfying $|L^c_{A,B}(x)| \ge \sqrt{\varepsilon}n^2$ is at least $2\sqrt{\varepsilon}n$ or the number of vertices $x \in J_A$ satisfying $|L_A(x)| \ge \sqrt{\varepsilon}n^2$ is at least $3\sqrt{\varepsilon}n$. If the first case holds, then we get $e^c(A,B) > \varepsilon n^3$. In the second case we have $e(A) > \varepsilon n^3$. Both are in contradiction with the choice of the partition $A \cup B$. Thus, $|J_A| \le 5\sqrt{\varepsilon}n$. The second statement of this claim, $|J_B| \le 5\sqrt{\varepsilon}n$, follows by a similar argument. ∎

Claim 6.6. *$A \setminus J_A$ and $B \setminus J_B$ are independent sets.*

Proof. If there is an edge $a_1a_2a_3$ with $a_1, a_2, a_3 \in A \setminus J_A$, since all its vertices satisfy $|L^c_B(a_i)| \le \sqrt{\varepsilon}n^2$, we can find a triangle in $L_B(a_1) \cap L_B(a_2) \cap L_B(a_3)$, call its vertices b_1, b_2, b_3. However, now $\{b_1, b_2, b_3, a_1, a_2, a_3\}$ spans an $F_{3,3}$ in H, a contradiction. A similar proof gives that $B \setminus J_B$ is an independent set. ∎

Claim 6.7. *There is no edge a_1, a_2, a_3 with $a_1 \in J_A$, $a_2, a_3 \in A \setminus J_A$ or with $a_1 \in J_B$, $a_2, a_3 \in B \setminus J_B$.*

Proof. Let a_1, a_2, a_3 be an edge of the type $a_1 \in J_A$, $a_2, a_3 \in A \setminus J_A$. The strategy of the proof is to show that $q(a_1) < d(n)$, to get a contradiction with (5). Let M_i, for $i = 2, 3$, be the set of non-edges in $L_B(a_i)$ and $L_{A,B}(a_i)$. Set $G = L(a_1) - M_2 - M_3$. Since $|M_2|, |M_3| \le 2\sqrt{\varepsilon}n^2$, we have $|E(G)| \ge |L(a_1)| - 4\sqrt{\varepsilon}n^2$. Let

$$\Delta = \frac{\max\limits_{x \in A \setminus \{a_1,a_2,a_3\}} |N_G(x) \cap B|}{n},$$

be the maximum size of a neighborhood in the graph G in B of a vertex in A, scaled by n. We have $0 \le \Delta \le |B|/n \le 1/2 + \sqrt{\varepsilon}$. Let $z \in A \setminus \{a_1, a_2, a_3\}$ such that $|N_G(z) \cap B)| = \Delta n$. Observe that $N_G(z) \cap B$ is an independent set in G, otherwise if $v, w \in N_G(z) \cap B$ with $vw \in E(G)$, then $\{v, w, z, a_1, a_2, a_3\}$ spans a $F_{3,3}$ in H. Now,

$$\sum_{x \in V \setminus \{a_1\}} d(a_1, x)^2 = \sum_{x \in V \setminus \{a_1\}} \deg_{L(a_1)}(x)^2 \le 16\sqrt{\varepsilon}n^3 + \sum_{x \in V(G)} \deg_G(x)^2, \tag{6}$$

because for each edge removed from the linkgraph $L(a_1)$ the degree squared sum can go down by at most $4n$. Now, we bound the sum on the right hand side of (6) from above. For $x \in A$, $\deg_G(x) \le |A| + \Delta n$ and for $x \in N(z) \cap B$, $\deg_G(x) \le n - \Delta n$. Thus, we get

$$\sum_{x \in V \setminus \{a_1\}} d(a_1, x)^2 \le 16\sqrt{\varepsilon} n^3 + |A|(|A| + \Delta n)^2 + \Delta n(n - \Delta n)^2 + (|B| - \Delta n)n^2$$

$$\le \left(\frac{n}{2} + 2\sqrt{\varepsilon} n\right)\left(\frac{n}{2} + 2\sqrt{\varepsilon} n + \Delta n\right)^2 + \Delta n(n - \Delta n)^2 + \left(\frac{n}{2} + 2\sqrt{\varepsilon} n - \Delta n\right) n^2 + 16\sqrt{\varepsilon} n^3$$

$$\le n^3 \left(\frac{1}{2}\left(\frac{1}{2} + \Delta\right)^2 + \Delta\,(1 - \Delta)^2 + \left(\frac{1}{2} - \Delta\right) + 25\sqrt{\varepsilon}\right) = n^3\left(\frac{5}{8} + \frac{\Delta}{2} - \frac{3}{2}\Delta^2 + \Delta^3 + 25\sqrt{\varepsilon}\right). \tag{7}$$

Furthermore, we can give an upper bound for the second sum in $q(a_1)$:

$$2\sum_{xy \in E(L(a_1))} d(x, y) \le 8\sqrt{\varepsilon} n^3 + 2\sum_{xy \in E(G)} d(x, y), \tag{8}$$

where we used that for each edge removed from G, the sum on the left hand side in (8) is lowered by at most n. Now, we will give an upper bound for the right hand side of (8). For edges $xy \in E(G[A])$ not incident to J_A we have $d_H(x, y) \le |J_A| + |B|$ because by Claim 6.6 they have no neighbor in $A \setminus J_A$. Similarly, for edges $xy \in E(G[B])$ not incident to J_B we have $d_H(x, y) \le |J_B| + |A|$. For all other edges $xy \in E(G)$, we will use the trivial bound $d_H(x, y) \le n$. We have

$$2\sum_{xy \in E(L(a_1))} d(x, y) \le 8\sqrt{\varepsilon} n^3 + 2\Big(e(G[A, B])n + e(G[A])(|J_A| + |B|) + |J_A||A|n$$
$$+ e(G[B])(|J_B| + |B|) + |J_B||B|n\Big). \tag{9}$$

By the choice of our partition we have $|L_A(x_1)| \le |L_B(x_1)|$ and thus $e(G[A]) \le e(G[B]) + 4\sqrt{\varepsilon} n^2$. Therefore, by upper bounding the right hand side in (9) we get

$$2\sum_{xy \in E(L(a_1))} d(x, y) \le 2\left(\Delta n^2|A| + 2e(G[B])\left(7\sqrt{\varepsilon} n + \frac{n}{2}\right) + 18\sqrt{\varepsilon} n^3\right)$$
$$\le 2n^3\left(\frac{\Delta}{2} + \frac{e(G[B])}{n^2} + 30\sqrt{\varepsilon}\right)$$
$$\le 2n^3\left(\frac{\Delta}{2} + \Delta\left(\frac{|B|}{n} - \Delta\right) + \frac{1}{4}\left(\frac{|B|}{n} - \Delta\right)^2 + 30\sqrt{\varepsilon}\right)$$
$$\le 2n^3\left(\frac{\Delta}{2} + \Delta\left(\frac{1}{2} - \Delta\right) + \frac{1}{4}\left(\frac{1}{2} - \Delta\right)^2 + 40\sqrt{\varepsilon}\right)$$
$$\le n^3\left(-\frac{3}{2}\Delta^2 + \frac{3}{2}\Delta + \frac{1}{8} + 80\sqrt{\varepsilon}\right), \tag{10}$$

where we used that $e(G[B]) \le \Delta n(|B| - \Delta n) + \frac{(|B| - \Delta n)^2}{4}$, because $G[B]$ contains an independent set of size Δn and is triangle-free. Now, we can combine (10) and (7) to upper bound $q(a_1)$.

$$q(a_1) \le n^3\left(\frac{5}{8} + \frac{\Delta}{2} - \frac{3}{2}\Delta^2 + \Delta^3 + 25\sqrt{\varepsilon}\right) + n^3\left(-\frac{3}{2}\Delta^2 + \frac{3}{2}\Delta + \frac{1}{8} + 80\sqrt{\varepsilon}\right)$$
$$= n^3\left(\Delta^3 - 3\Delta^2 + 2\Delta + \frac{3}{4} + 105\sqrt{\varepsilon}\right) \le (1.14 + 97\sqrt{\varepsilon})n^3 < \frac{5}{4}n^3 - 6n^2,$$

contradicting (5). ∎

Now, we can make use of Claim 6.7 to show that there is no edge inside A, respectively inside B.

Claim 6.8. *A and B are independent sets.*

Proof. Let $\{a_1, a_2, a_3 \in A\}$ span an edge. Again, $L_B(a_1) \cap L_B(a_2) \cap L_B(a_3)$ is triangle-free. Thus, $|L_B(a_1) \cap L_B(a_2) \cap L_B(a_3)| \leq |B|^2/4$. Without loss of generality, we have $|L_B(a_1)| \leq 5|B|^2/12$. Furthermore, by Claims 6.6 and 6.7, $|L_A(a_1)| \leq |J_A||A| \leq 5\sqrt{\varepsilon}n^2$. Again, our strategy will be to give an upper bound on $q(a_1)$. Let G be the graph obtained from $L(a_1)$ when all edges inside A are being removed.

$$\sum_{x \in V \setminus \{a_1\}} d(a_1, x)^2 = \sum_{x \in V \setminus \{a_1\}} \deg_{L(a_1)}(x)^2 \leq 20\sqrt{\varepsilon}n^3 + \sum_{x \in V(G)} \deg_G(x)^2$$
$$\leq 20\sqrt{\varepsilon}n^3 + |B|n^2 + |A||B|^2 \leq n^3 \left(\frac{5}{8} + 30\sqrt{\varepsilon}\right). \tag{11}$$

Furthermore,

$$2 \sum_{xy \in E(L(a_1))} d(x, y) \leq 10\sqrt{\varepsilon}n^3 + 2 \sum_{xy \in E(G)} d(x, y)$$
$$\leq 2\left(\frac{5}{12}|B|^2 (|A| + |J_B|) + 10\sqrt{\varepsilon}n^3 + |A||B|n\right)$$
$$\leq 2n^3 \left(\frac{5}{96} + 20\sqrt{\varepsilon} + \frac{1}{4}\right) = n^3 \left(\frac{29}{48} + 40\sqrt{\varepsilon}\right). \tag{12}$$

Thus, by combining (11) and (12), we can give an upper bound on $q(a_1)$,

$$q(a_1) \leq \left(\frac{5}{8} + 30\sqrt{\varepsilon}\right) n^3 + n^3 \left(\frac{29}{48} + 40\sqrt{\varepsilon}\right) = n^3 \left(\frac{59}{48} + 70\sqrt{\varepsilon}\right) < \frac{5}{4}n^3 - 6n^2,$$

contradicting (5). Therefore A is an independent set. By a similar argument B is also an independent set.

∎

By Claim 6.8, H is 2-colorable. Since among all 2-colorable 3-graphs B_n has the largest codegree squared sum, we conclude $\text{co}_2(H) \leq \text{co}_2(B_n)$. This completes the proof of Theorem 6.3. ∎

We now complete the proof of Theorem 6.3 by showing that imposing the additional assumption (5) is not more restrictive.

Proof of Theorem 1.7. Let H be an n-vertex 3-uniform $F_{3,3}$-free hypergraph which has a codegree squared sum of at least $\text{co}_2(H) \geq \text{co}_2(B_n)$. Set $d(n) = 5/4n^3 - 6n^2$ and note that $\text{co}_2(B_n) - \text{co}_2(B_{n-1}) > d(n) + 1$. We claim that we can assume that every vertex $x \in V(H)$ satisfies (5). Otherwise, we can remove a vertex x with $q(x) < d(n)$ to get H_{n-1} with $\text{co}_2(H_{n-1}) \geq \text{co}_2(B_n) - d(n) \geq \text{co}_2(B_{n-1}) + 1$. By repeating this process as long as possible, we obtain a sequence of hypergraphs H_m on m vertices with $\text{co}_2(H_m) \geq \text{co}_2(B_m) + n - m$, where H_m is the hypergraph obtained from H_{m+1} by deleting a vertex x with $q(x) \leq d(m+1)$. We cannot continue until we reach a hypergraph on $n_0 = n^{1/4}$ vertices, as then $\text{co}_2(H_{n_0}) > n - n_0 > \binom{n_0}{2}(n_0 - 2)^2$ which is impossible. Therefore, the process stops at some n' where $n \geq n' \geq n_0$ and we obtain the corresponding hypergraph $H_{n'}$ satisfying $q(x) \geq d(n')$ for all $x \in V(H_{n'})$ and $\text{co}_2(H_{n'}) \geq \text{co}_2(B_{n'})$ (with strict inequality if $n > n'$). Hence, we can assume that H satisfies $q(x) \geq d(n')$ for all $x \in V(H_{n'})$. Applying Theorem 6.3 finishes the proof. ∎

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