On tripartite common graphs

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Abstract

A graph H is *common* if the number of monochromatic copies of H in a 2-edge-colouring of the complete graph K_n is minimised by the random colouring. Burr and Rosta, extending a famous conjecture by Erdős, conjectured that every graph is common. The conjectures by Erdős and by Burr and Rosta were disproved by Thomason and by Sidorenko, respectively, in the late 1980s. Collecting new examples for common graphs had not seen much progress since then, although very recently, a few more graphs are verified to be common by the flag algebra method or the recent progress on Sidorenko's conjecture.

Our contribution here is to give a new class of tripartite common graphs. The first example class is so-called triangle-trees, which generalises two theorems by Sidorenko and answers a question by Jagger, Šťovíček, and Thomason from 1996. We also prove that, somewhat surprisingly, given any tree T, there exists a triangle-tree such that the graph obtained by adding T as a pendant tree is still common. Furthermore, we show that adding arbitrarily many apex vertices to any connected bipartite graph on at most 5 vertices give a common graph.

1 Introduction

Ramsey's theorem states that, for a fixed graph H, every 2-edge-colouring of K_n contains a monochromatic copy of H whenever n is large enough. Perhaps one of the most natural questions after Ramsey's theorem is then how many copies of monochromatic H can be guaranteed to exist. To formalise this question, let the *Ramsey multiplicity* $M(H;n)$ be the minimum number of labelled copies of monochromatic H over all 2-edge-colouring of K_n. We define the *Ramsey multiplicity constant* $C(H)$ is defined by

$$C(H) := \lim_{n\to\infty} \frac{M(H,n)}{n(n-1)\cdots(n-v+1)} = \lim_{n\to\infty} M(H,n)\cdot n^{-v},$$

where v is the number of vertices in H. One may easily check that $C(H) \leq 2^{1-e(H)}$ by considering the random colouring of K_n. A graph is *common* if $C(H) = 2^{1-e(H)}$. For example, Goodman's formula [11] implies that a triangle is common, i.e., $C(K_3) = 1/4$.

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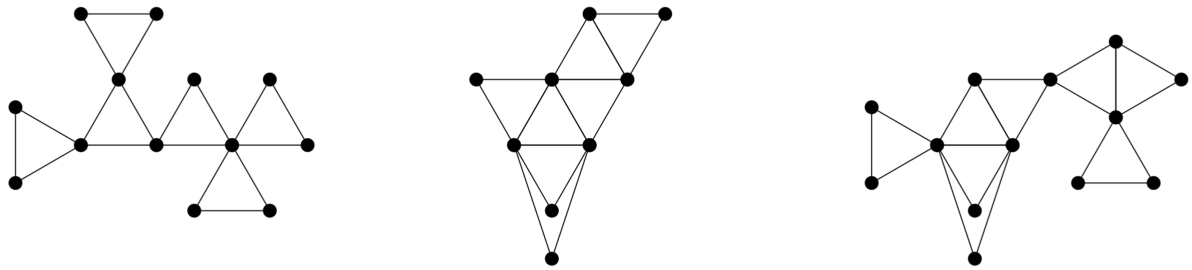


Triangle-vertex-tree Triangle-edge-tree Triangle-tree

Figure 1: Triangle-vertex-tree, triangle-edge-tree, and triangle-tree.

In 1962, Erdős [8] conjectured that every complete graph K_t is common. This was later generalised by Burr and Rosta [4], who conjectured that in fact every graph H is common. In the late 1980s, both conjectures were disproved. Sidorenko [23] gave an example, a triangle plus a pendant edge, of an uncommon graph, and Thomason [28] proved that every K_t, $t \geq 4$, is uncommon.

Since then, more examples of uncommon graphs have been found. For instance, Jagger, Šťovíček and Thomason [14] proved that every graph containing K_4 as a subgraph is uncommon, and Fox [9] proved that $C(H)$ can be exponentially smaller than the commonality bound $2^{1-e(H)}$.

Despite its importance, the full classification of common graphs is still a wide open problem. Finding bipartite common graphs relies entirely on the recent progress on Sidorenko's conjecture [24], since the conjecture implies that every bipartite graph is common. The converse is however unknown to be true or not, although very recently it was shown [15] that a bipartite graph satisfies Sidorenko's conjecture if and only if it is common in *any* multi-colour sense. There have been some progress on Sidorenko's conjecture (see, for example, [6] and references therein) but the full conjecture remains open.

For non-bipartite common graphs, there are fewer known examples. One of the earliest applications of the flag algebra method in [13] proved that the 5-wheel is common. For tripartite graphs, a few more examples have been collected, e.g., odd cycles [23] and even wheels [14, 25].

Two of very few general example classes of non-bipartite common graphs are *triangle-vertex-trees* and *triangle-edge-trees*, obtained by Sidorenko [25] and reproved by Jagger, Šťovíček, and Thomason [14]. These can be described recursively. A single triangle is a *triangle-tree* and one may obtain a triangle-tree by identifying a single vertex or an edge of a new triangle with a vertex or an edge, respectively, in a triangle-tree. A triangle-tree is a triangle-vertex-tree (resp. triangle-edge-tree) if it is obtained by identifying only vertices (resp. edges). See Figure 1 for examples.

Jagger, Šťovíček and Thomason [14] asked a question to decide whether other tree-like structures than triangle-vertex (or triangle-edge) trees formed from triangles are common. In particular, they asked if a triangle-tree formed by three triangles, as described in Figure 2, is common. We ultimately answer these questions.

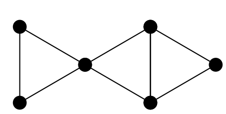


Figure 2: A triangle-tree questioned by Jagger, Šťovíček and Thomason [14].

Theorem 1.1. *Every triangle-tree is common.*

As implicitly evidenced by fewer known examples, non-bipartite graphs are more 'likely' to be uncommon than bipartite graphs in two ways. Firstly, no examples of bipartite uncommon graphs, which will disprove Sidorenko's conjecture if exist, are known. Second, there is a well-known strategy to produce a non-bipartite uncommon graphs. That is, adding a (possibly large) pendant tree, e.g., a long path.

Sidorenko's counterexample, the triangle plus a pendant edge, for the Burr–Rosta conjecture can be seen as one of the earliest examples of such kind. For another example, Fox [9, Lemma 2.1] observed that a graph with chromatic number at least four and small enough average degree is always uncommon. As a corollary, adding a long pendant path to substantially lower the average degree of the graph always guarantees that the resulting graph is uncommon. However, the same strategy does not apply straightforwardly to tripartite graphs. Fox instead proved in the same paper that the complete tripartite graph $K_{t,t,t}$ plus a pendant path of length $\Omega(t^2)$ is uncommon, but, at the best of our knowledge, there is no general statement that can be compared to the higher chromatic number case.

Our second result gives a reason why adding pendant trees to tripartite graphs is not as powerful as the case for graphs with a higher chromatic number in constructing uncommon graphs. For a tree T and a graph H, let $T *_u^v H$ be the graph obtained by identifying $u \in V(T)$ and $v \in V(H)$.

Theorem 1.2. *Let t be a positive integer. Then there exists a tripartite common graph H such that $T *_u^v H$ is common for every choice of tree T with $e(T) \leq t$, $u \in V(T)$, and $v \in V(H)$.*

In other words, there is a tripartite graph H that is 'robustly common' in the sense that adding any tree T of bounded size does not break its commonality.

At the Canadian Discrete and Algorithmic Mathematics Conference in 2017, Julia Wolf asked a question to complete the list of connected common graphs on five vertices during her plenary talk on [22] that concerns with arithmetic variants of commonality questions. As a corollary, we prove commonality of the two graphs in Figure 3 that were unknown to be common by then; see Concluding Remarks for more discussions about Wolf's list.

Figure 3: Example applications of Theorem 1.2.

Another interesting class of tripartite common graphs was obtained by Sidorenko [25] built upon his conjecture. If a connected bipartite graph H satisfies Sidorenko's conjecture, then adding an *apex* vertex v, i.e., adding all the edges between the new vertex v and each vertex of H, gives a tripartite common graph. We conjecture that adding more apex vertices still produces common graphs. For a graph H and a positive integer a, let H^{+a} be the graph obtained from H by adding additional a vertices fully connected to H and not connected to each other.

Conjecture 1.3. *If a connected bipartite graph H satisfies Sidorenko's conjecture, then for every positive integer a the graph H^{+a} is common. In particular, every complete tripartite graph $K_{r,s,t}$ is common.*

We verify this conjecture for all connected bipartite graphs H on at most 5 vertices, so, in particular, the complete tripartite graphs $K_{2,2,a}$ and $K_{2,3,a}$ are common for every $a \geq 1$.

Theorem 1.4. *For every connected bipartite graph H on at most 5 vertices and positive integer a the graph H^{+a} is common.*

The proof of Theorem 1.4 relies on the computer-assisted flag algebra method, but we also give a computer-free proof for some cases. In particular, we prove that the octahedron graph, i.e., $C_4^{+2} = K_{2,2,2}$, is common without using computers, which will then generalise to the so-called beachball graphs (see Theorem 4.3).

2 Preliminaries

A *graph homomorphism* from a graph H to a graph G is a vertex map that preserves adjacency. Let $\mathrm{Hom}(H, G)$ denote the set of all homomorphisms from H to G and let $t_H(G)$ be the probability that a uniform random mapping from H to G is a homomorphism. i.e., $t_H(G) = \frac{|\mathrm{Hom}(H,G)|}{|V(G)|^{|V(H)|}}$.

The graph homomorphism density $t_H(G)$ naturally extends to weighted graphs and their limit object *graphons*, i.e., measurable symmetric functions $W : [0, 1]^2 \to [0, 1]$. We define

$$t_H(W) := \mathbb{E}\left(\prod_{uv\in E(H)} W(x_u, x_v)\right),$$

where $\mathbb{E}$ denotes the integration with respect to the Lebesgue measure on $[0, 1]^{V(H)}$. One may see that the original definition of $t_H(G)$ corresponds to the case $W = W_G$, where W_G is the block 0-1 graphon constructed by the adjacency matrix of G. As nonnegativity of W is unnecessary for the definition, we shall also use $t_H(U) := \mathbb{E}\left(\prod_{uv\in E(H)} U(x_u, x_v)\right)$ for *signed graphons* U, i.e., measurable symmetric functions $U : [0, 1]^2 \to [-1, 1]$.

The monochromatic (labelled) copies of H in a 2-edge-colouring can be rewritten as

$$m_H(W) := t_H(W) + t_H(1 - W).$$

Note that $m_H(W) = m_H(1-W)$ and $C(H) = \min_W m_H(W)$, where the minimum is taken over all graphons W. Indeed, the minimum exists by the compactness of the space of graphon under the cut norms and the latter follows from the standard W-random graph sampling (see, for example, [17]). Thus, a graph H is common if and only if $m_H(W) \geq 2^{1-e(H)}$ for each graphon W.

Let $\mathcal{E}(H)$ be the family of subgraphs of H with even number of edges and let $\mathcal{E}_+(H)$ be the collection of nonempty graphs in $\mathcal{E}(H)$. Then, with $U := 2W - 1$,

$$\begin{aligned} m_H(W) &= t_H\left(\frac{1+U}{2}\right) + t_H\left(\frac{1-U}{2}\right) \\ &= 2^{1-e(H)} \sum_{F\in\mathcal{E}(H)} t_F(U) = 2^{1-e(H)}\left(1 + \sum_{F\in\mathcal{E}_+(H)} t_F(U)\right). \end{aligned} \tag{1}$$

Hence, H is common if and only if $\sum_{F\in\mathcal{E}_+(H)} t_F(U) \geq 0$ for every signed graphon U.

An immediate consequence of this expansion is a well-known formula by Goodman [11].

Lemma 2.1 (Goodman's formula)**.** *For every graphon W, $m_{K_3}(W) = \frac{3}{2}m_{K_{1,2}}(W) - \frac{1}{2}$.*

Proof. By (1), $m_{K_3}(W) = \frac{3}{4}t_{K_{1,2}}(U) + \frac{1}{4}$ and $m_{K_{1,2}}(W) = \frac{1}{2}t_{K_{1,2}}(U) + \frac{1}{2}$. □

The following is an easy consequence of Hölder's inequality, which will be repeatedly used.

Lemma 2.2. *Let H, F, and J be graphs and let k and ℓ be positive integers with $\ell \geq k$. For a graphon W, if*

$$t_H(W) \geq \frac{t_J(W)^\ell}{t_F(W)^{k-1}} \quad \text{and} \quad t_H(1-W) \geq \frac{t_J(1-W)^\ell}{t_F(1-W)^{k-1}},$$

then

$$m_H(W) \geq 2^{k-\ell} \frac{m_J(W)^\ell}{m_F(W)^{k-1}}.$$

Proof. We use Hölder's inequality of the form

$$\prod_{i=1}^{k} \int f_i(x)^k dx \geq \left(\int \prod_{i=1}^{k} f_i(x) dx \right)^k$$

for nonnegative functions f_i. Let the integration be the sum of two terms. Then

$$\prod_{i=1}^{k} (a_i^k + b_i^k) \geq \left(\prod_{i=1}^{k} a_i + \prod_{j=1}^{k} b_j \right)^k \tag{2}$$

for nonnegative numbers a_i and b_j, it follows that

$$\begin{aligned} m_H(W) = t_H(W) + t_H(1-W) &\geq \frac{t_J(W)^\ell}{t_F(W)^{k-1}} + \frac{t_J(1-W)^\ell}{t_F(1-W)^{k-1}} \\ &= \left(\frac{t_J(W)^\ell}{t_F(W)^{k-1}} + \frac{t_J(1-W)^\ell}{t_F(1-W)^{k-1}} \right) \left(t_F(W) + t_F(1-W) \right)^{k-1} m_F(W)^{-k+1} \\ &\geq_{(2)} \left(t_J(W)^{\frac{\ell}{k}} + t_J(1-W)^{\frac{\ell}{k}} \right)^k m_F(W)^{-k+1} \\ &\geq 2^{k-\ell} \frac{m_J(W)^\ell}{m_F(W)^{k-1}}. \end{aligned}$$

Indeed, the first inequality is Hölder's inequality (2) and the second follows from convexity of the function $f(z) = z^{\ell/k}$, as $\ell \geq k$. □

For the proof of Theorem 1.2, we take an information-theoretic approach. We state the following fact about entropy without proofs and refer the reader to [1] for more detailed information on entropy and conditional entropy.

Lemma 2.3. *Let X, Y, and Z be random variables and suppose that X takes values in a set S, $\mathbb{H}(X)$ is the entropy of X. Then $\mathbb{H}(X) \leq \log |S|$.*

3 Triangle-trees

To describe triangle-trees, it is convenient to use the notion of tree decompositions, introduced by Halin [12] and developed by Robertson and Seymour [21].

Definition 3.1. A *tree-decomposition* of a graph H is a pair $(\mathcal{F}, \mathcal{T})$ consisting of a family $\mathcal{F}$ of vertex subsets of H and a tree $\mathcal{T}$ with $V(\mathcal{T}) = \mathcal{F}$ such that

1. $\bigcup_{X \in \mathcal{F}} X = V(H)$,

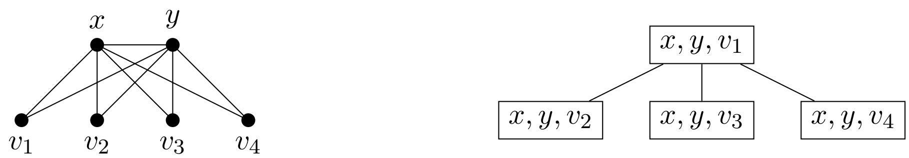


Figure 4: $K_{1,1,4}$ and its tree decomposition $(\mathcal{F}, \mathcal{T})$.

2. for each $e \in E(H)$, there exists a set $X \in \mathcal{F}$ such that X contains e, and
3. for $X, Y, Z \in \mathcal{F}$, $X \cap Y \subseteq Z$ whenever Z lies on the path from X to Y in $\mathcal{T}$.

Following [7, 16], we say that H is a *J-tree* if and only if there exists a tree decomposition $(\mathcal{F}, \mathcal{T})$ such that the subgraph $H[X]$ of H induced on $X \in \mathcal{F}$ is isomorphic to J and moreover, there is an isomorphism between $H[X]$ and $H[Y]$ that fixes $H[X \cap Y]$ whenever $XY \in E(\mathcal{T})$. Such a tree-decomposition $(\mathcal{F}, \mathcal{T})$ of H is called a *J-decomposition.* When $J = K_3$, we simply say that H is a triangle-tree with a triangle-decomposition $(\mathcal{F}, \mathcal{T})$. It is straightforward to see that this definition is equivalent to the recursive one given in the introduction.

For a triangle-tree H with a triangle-decomposition $(\mathcal{F}, \mathcal{T})$, one may easily relate $|\mathcal{F}|$ to $v(H)$ and $e(H)$. Let $\varphi(H) := e(H) - v(H) + 1$ and $\kappa(H) := 2e(H) - 3v(H) + 3$.

Lemma 3.2. *If H is a triangle-tree with a triangle-decomposition $(\mathcal{F}, \mathcal{T})$, then $|\mathcal{F}| = \varphi(H)$ and the number of edges $XY \in E(\mathcal{T})$ such that the subgraph $H[X \cap Y]$ is a single edge equals to $\kappa(H)$. In particular, $\kappa(H) \leq \varphi(H) - 1$ for every triangle-tree H.*

Proof. Let $k := k(\mathcal{F})$ be the number of edges $XY \in E(\mathcal{T})$ such that the subgraph $H[X \cap Y]$ is an edge. For an edge $e \in E(H)$, let t_e be the number of contributions of e in the sum $\sum_{X \in \mathcal{F}} e(H[X])$. That is,

$$3|\mathcal{F}| = \sum_{X \in \mathcal{F}} e(H[X]) = \sum_{e \in E(H)} t_e.$$

On the other hand, $t_e - 1$ is equal to the number of edges $XY \in E(\mathcal{T})$ such that $H[X \cap Y]$ is the single-edge $\{e\}$, which proves $e(H) = 3|\mathcal{F}| - k$. Analogously, $v(H) = 2|\mathcal{F}| + 1 - k$ and hence, $|\mathcal{F}| = e(H) - v(H) - 1 = \varphi(H)$ and $k(\mathcal{F}) = 2e(H) - 3v(H) + 3 = \kappa(H)$. Finally, $\kappa(H) = k \leq e(\mathcal{T}) = |\mathcal{F}| - 1 = \varphi(H) - 1$. □

The key ingredient in the proof of Theorem 1.1 is the following lemma.

Lemma 3.3 ([16], Theorem 2.7)**.** *Let H be a J-tree with a J-decomposition $(\mathcal{F}, \mathcal{T})$ and let W be a nonzero graphon. Then*

$$t_H(W) \geq \frac{t_J(W)^{|\mathcal{F}|}}{\prod_{XY \in E(\mathcal{T})} t_{H[X \cap Y]}(W)}. \tag{3}$$

Lemma 3.3 in particular reproves standard applications of the Cauchy–Schwarz inequality or Jensen's inequality. For example, $K_{1,1,t}$ is a triangle-tree, since there is a triangle decomposition $(\mathcal{F}, \mathcal{T})$ that consists of $|\mathcal{F}| = t$ and the star $\mathcal{T}$ on $\mathcal{F}$ with $t - 1$ leaves, where each vertex subset in $\mathcal{F}$ induces a triangle; see Figure 4. Thus, Lemma 3.3 gives $t_{K_{1,1,t}}(W) \geq t_{K_3}(W)^t / t_{K_2}(G)^{t-1}$, which also follows from a standard application of Jensen's inequality.

In order to prove Theorem 1.1, we are going to apply Lemma 3.3 for $J = K_3$.

Corollary 3.4. *If H is a triangle-tree and W is a nonzero graphon, then*

$$t_H(W) \geq \frac{t_{K_3}(W)^{\varphi(H)}}{t_{K_2}(W)^{\kappa(H)}}. \tag{4}$$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let H be a triangle-tree. If $W = 1$ or $W = 0$ almost everywhere, then $m_H(W) = 1$. Otherwise, two applications of (4) yields

$$t_H(W) \geq \frac{t_{K_3}(W)^{\varphi(H)}}{t_{K_2}(W)^{\kappa(H)}} \quad \text{and} \quad t_H(1-W) \geq \frac{t_{K_3}(1-W)^{\varphi(H)}}{t_{K_2}(1-W)^{\kappa(H)}}.$$

Therefore, by Lemma 2.2 with $J = K_3$, $H = K_2$, $\ell = \varphi(H)$ and $k = \kappa(H) + 1$, we have

$$m_H(W) \geq 2^{\kappa(H)+1-\varphi(H)} \cdot \frac{m_{K_3}(W)^{\varphi(H)}}{m_{K_2}(W)^{\kappa(H)}} = 2^{\kappa(H)+1-\varphi(H)} \cdot m_{K_3}(W)^{\varphi(H)} \geq 2^{\kappa(H)+1-3\varphi(H)} = 2^{1-e(H)}.$$

Indeed, the last inequality uses the commonality of a triangle, i.e., $m_{K_3}(W) \geq 1/4$. □

To prove Theorem 1.2, we need a slightly more careful analysis than just a simple application of Lemma 3.3. The main tool is [16, Theorem 2.6], which will be stated shortly. Let $\mathcal{F}$ be a family of subsets of $[k] := \{1, 2, \ldots, k\}$. A *Markov tree* on $[k]$ is a pair $(\mathcal{F}, \mathcal{T})$ with $\mathcal{T}$ a tree on vertex set $\mathcal{F}$ that satisfies

1. $\bigcup_{F \in \mathcal{F}} F = [k]$ and
2. for $A, B, C \in \mathcal{F}$, $A \cap B \subseteq C$ whenever C lies on the path from A to B in $\mathcal{T}$.

This is an abstract tree-like structure without the graph structure considered in defining tree-decompositions. In particular, a tree-decomposition of H is a Markov tree on $V(H)$. For more detailed explanation, we refer to [16]. Let V be a finite set and for each $F \in \mathcal{F}$ let $\mathbf{X}_F = (X_{i;F})_{i \in F}$ be a random vector taking values in V^F. The following theorem states that there exist random variables $Y_1, Y_2, \ldots, Y_k$ such that, for each $F \in \mathcal{F}$, the two random vectors $(Y_i)_{i \in F}$ and $\mathbf{X}_F$ are identically distributed over V^F and, moreover, the maximum entropy under such constraints can always be attained.

Lemma 3.5 ([16], Theorem 2.6)**.** *Let $(\mathcal{F}, \mathcal{T})$ be a Markov tree on $[k]$. Let V be a finite set and for each $F \in \mathcal{F}$ let $\mathbf{X}_F = (X_{i;F})_{i \in F}$ be a random vector taking values in V^F. If $(X_{i;A})_{i \in A \cap B}$ and $(X_{j;B})_{j \in A \cap B}$ are identically distributed whenever $AB \in E(\mathcal{T})$, then there exists $\mathbf{Y} = (Y_1, \ldots, Y_k)$ with entropy*

$$\mathbb{H}(\mathbf{Y}) = \sum_{F \in \mathcal{F}} \mathbb{H}(\mathbf{X}_F) - \sum_{AB \in E(\mathcal{T})} \mathbb{H}((X_{i;A})_{i \in A \cap B}) \tag{5}$$

such that $(Y_i)_{i \in F}$ and $\mathbf{X}_F$ are identically distributed over V^F for all $F \in \mathcal{F}$.

An entropy analysis using this lemma give the following corollary. Recall that for a tree T and a graph H, we denote by $T *_u^v H$ the graph obtained by identifying $u \in V(T)$ and $v \in V(H)$.

Lemma 3.6. *If H is a triangle-tree and T a tree with at most $\kappa(H)$ edges, then*

$$t_{T *_u^v H}(W) \geq \frac{t_{K_3}(W)^{\varphi(H)}}{t_{K_2}(W)^{\kappa(H)-e(T)}} \tag{6}$$

for every $u \in V(T)$ and $v \in V(H)$.

Using this lemma, the proof of Theorem 1.2 is almost identical to that of Theorem 1.1.

Proof of Theorem 1.2. Let H be a triangle-tree such that $\kappa(H) \geq t$, T a tree with at most t edges, and W a nonzero graphon. By Lemma 3.2, $e(H) = 3\varphi(H) - \kappa(H)$ and thus,

$$e(T *_u^v H) = e(T) + e(H) = 3\varphi(H) - \kappa(H) + e(T).$$

Combining (6) and Lemma 2.2 for $J = K_3$, $H = K_2$, $\ell = \varphi(H)$ and $k = \kappa(H) - e(T) + 1$ yields

$$m_{T *_u^v H}(W) \geq 2^{\kappa(H)+1-e(T)-\varphi(H)} \cdot m_{K_3}(W)^{\varphi(H)}.$$

Note that in order to apply Lemma 2.2, we required $\kappa(H) \geq e(T)$. As $m_{K_3}(W) \geq 1/4$, we have

$$m_{T *_u^v H}(W) \geq 2^{\kappa(H)+1-e(T)-3\varphi(H)} = 2^{1-e(H)-e(T)} = 2^{1-e(T *_u^v H)}.$$

Therefore, $T *_u^v H$ is common. □

It remains to prove Lemma 3.6.

Proof of Lemma 3.6. Let $(\mathcal{F}, \mathcal{T})$ be a triangle-decomposition of H and $k := \kappa(H)$. Recall that k is the number of edges $XY \in E(\mathcal{T})$ such that the subgraph $H[X \cap Y] \cong K_2$. The first step is to find a natural tree-decomposition of $T *_u^v H$ that extends $(\mathcal{F}, \mathcal{T})$.

Let T be rooted at a leaf $x \in V(T)$ and suppose that we orient each edge of T away from the root. Let $\mathcal{S}$ be a tree on $E(T)$, where the oriented edges (u_1, v_1) and (u_2, v_2) are adjacent if and only if $v_1 = u_2$. One may easily check that $(E(T), \mathcal{S})$ is a tree decomposition of T. Now pick an edge $uu' \in E(T)$, which is a vertex of $\mathcal{S}$, and connect it to a vertex bag $X \in \mathcal{F}$ that contains $v \in V(T)$ while identifying u and v. This new tree $\mathcal{T}'$, obtained by adding an edge between two vertices uu' and X, gives a tree-decomposition $(\mathcal{F}', \mathcal{T}')$ of $T *_u^v H$, where $\mathcal{F}' := V(\mathcal{T}') = V(\mathcal{T}) \cup V(\mathcal{S})$.

It is enough to prove the inequality (6) for an n-vertex graph G instead of a graphon W, since there is a sequence of graphs $(G_i)_{i=1}^{\infty}$ that 'converges' to W, i.e., $t_J(G_i) \to t_J(W)$ as $i \to \infty$ for every fixed graph J. For brevity, we identify the vertex set $V(T *_u^v H)$ with the set $[t]$ and let $1 \in [t]$ be the vertex shared by H and T. For each $F \in \mathcal{F}'$ with $|F| = 3$, let $\mathbf{X}_F = (X_{i;F})_{i \in F}$ be a uniform random triangle in G, labelled by vertices in F. If $|F| = 2$ then let $\mathbf{X}_F = (X_{i;F})_{i \in F}$ be a random edge labelled by vertices in F sampled in such a way that $\mathbb{P}[\mathbf{X}_F = (v_1, v_2)]$ is proportional to the number of triangles that contains the edge $v_1 v_2 \in E(G)$. We call this possibly non-uniform edge distribution *triangle-projected*.

We claim that $(X_{i;A})_{i \in A \cap B}$ and $(X_{i;B})_{i \in A \cap B}$ are identically distributed. If $|A \cap B| = 2$, then both distributions are triangle-projected. If $|A \cap B| = 1$, i.e., $A \cap B = x \in V(T *_u^v H)$, then both distributions are proportional to the weighted degree sum $\sum_{x \subset e} p_e$ where p_e is the probability of an edge being sampled by the triangle-projected distribution. Therefore, by Lemma 3.5, there exists $\mathbf{Y} = (Y_1, \ldots, Y_t)$ with entropy

$$\begin{aligned}
\mathbb{H}(\mathbf{Y}) &= \sum_{F \in \mathcal{F}'} \mathbb{H}(\mathbf{X}_F) - \sum_{AB \in E(\mathcal{T}')} \mathbb{H}((X_{i;A})_{i \in A \cap B}) \\
&= \sum_{F \in \mathcal{F}} \mathbb{H}(\mathbf{X}_F) + \sum_{F \in E(T)} \mathbb{H}(\mathbf{X}_F) - \sum_{AB \in E(\mathcal{T})} \mathbb{H}((X_{i;A})_{i \in A \cap B}) - \sum_{AB \in E(\mathcal{S})} \mathbb{H}((X_{i;A})_{i \in A \cap B}) - \mathbb{H}(Y_1).
\end{aligned}$$

Recall that the vertex 1 is the vertex shared by T and H, so Y_1 means the random image of the vertex with respect to $\mathbf{Y}$. For $F \in \mathcal{F}$, $\mathbb{H}(\mathbf{X}_F) = \log |\mathrm{Hom}(K_3, G)|$, since $\mathbf{X}_F$ is a uniform random triangle. For $F \in E(T)$, $\mathbb{H}(\mathbf{X}_F)$ is the entropy h_e of the triangle-projected edge distribution. There are exactly k cases such that $|A\cap B|=2$ and $AB\in E(\mathcal{T})$, and for such cases, $\mathbb{H}((X_{i;A})_{i\in A\cap B})=h_e$. Thus,

$$\begin{aligned}\mathbb{H}(\mathbf{Y}) &= |\mathcal{F}|\log|\mathrm{Hom}(K_3,G)| + \sum_{F\in E(T)}\mathbb{H}(\mathbf{X}_F) - \sum_{AB\in E(\mathcal{T}')}\mathbb{H}((X_{i;A})_{i\in A\cap B}) \\ &\geq |\mathcal{F}|\log|\mathrm{Hom}(K_3,G)| - (k-e(T))h_e - (e(\mathcal{T}')-k)\log|V(G)| \\ &\geq |\mathcal{F}|\log|\mathrm{Hom}(K_3,G)| - (k-e(T))\log|\mathrm{Hom}(K_2,G)| - (e(\mathcal{T}')-k)\log n\end{aligned}$$

Indeed, the first inequality follows from the bound $\mathbb{H}((X_{i;A})_{i\in A\cap B})\leq \log n$ by Lemma 2.3 when $|A\cap B|=1$, and the second follows from the bound $h_e\leq \log|\mathrm{Hom}(T*_u^v H,G)|$ by the same lemma. Again by Lemma 2.3, $\mathbb{H}(\mathbf{Y})\leq \log|\mathrm{Hom}(H,G)|$. Thus,

$$\begin{aligned}t_{T*_u^v H}(G) &= \frac{|\mathrm{Hom}(T*_u^v H,G)|}{n^{v(H)+v(T)-1}} \geq \frac{|\mathrm{Hom}(K_3,G)|^{|\mathcal{F}|}}{|\mathrm{Hom}(K_2,G)|^{k-e(T)}n^{e(\mathcal{T}')-k}}\cdot\frac{1}{n^{v(H)+v(T)-1}} \\ &= \frac{t_{K_3}(G)^{|\mathcal{F}|}}{t_{K_2}(G)^{k-e(T)}n^{e(\mathcal{T}')+k-2e(T)}}\cdot\frac{n^{3|\mathcal{F}|}}{n^{v(H)+v(T)-1}} = \frac{t_{K_3}(G)^{|\mathcal{F}|}}{t_{K_2}(G)^{k-e(T)}},\end{aligned}$$

where the last equality follows from the identity $e(\mathcal{T}')=e(\mathcal{T})+e(\mathcal{S})+1=|\mathcal{F}|+e(T)-1$ and Lemma 3.2. □

4 Beachball graphs and bipartite graphs with apex vertices

The proof of Theorem 1.4 combines our novel ideas and the flag algebra method developed by Razborov [20]. To demonstrate how the new method works without using flag algebras, we firstly prove that $K_{2,2,2}$ is common.

Theorem 4.1. *The octahedron $K_{2,2,2}$ is common.*

By a standard application of the Cauchy–Schwarz inequality (or Lemma 3.3), it is easy to see that $t_{K_{2,2,2}}(W)\geq t_{K_{1,2,2}}(W)^2/t_{C_4}(W)$. Then by Lemma 2.2, we immediately obtain

$$m_{K_{2,2,2}}(W)\geq \frac{m_{K_{1,2,2}}(W)^2}{m_{C_4}(W)}. \tag{7}$$

By Sidorenko's theorem [25], the 4-wheel $K_{1,2,2}$ is common; however, $m_{C_4}(W)=1/8$ if and only if $W=1/2$ almost everywhere, i.e., W is quasirandom, the naive approach using commonality of $K_{1,2,2}$ while bounding m_{C_4} from above does not work. We circumvent this difficulty by comparing $m_{K_{1,2,2}}(W)$ and $m_{C_4}(W)$. Another application of the Cauchy–Schwarz inequality, together with Lemma 2.2, gives

$$m_{K_{1,2,2}}(W)\geq \frac{m_{K_{1,1,2}}(W)^2}{m_{K_{1,2}}(W)}.$$

For brevity, denote $D:=K_{1,1,2}$, which is the *diamond* graph obtained by adding a diagonal edge to the 4-cycle. The following lemma, partly motivated by [13], enables us to compare $m_D(W)$ and $m_{C_4}(W)$.

Lemma 4.2. *Let $0\leq c\leq (3-\sqrt{5})/4$. For any graphon W, the following inequality holds:*

$$m_D(W)-1/16\geq c(m_{C_4}(W)-1/8).$$

Proof. Using (1) with $U := 2W - 1$ and $H = D$, we obtain

$$m_D(W) - \frac{1}{16} = \frac{1}{16} \sum_{F \in \mathcal{E}_+(D)} t_F(U) = \frac{1}{16}\big(2t_{2\cdot K_2}(U) + 8t_{K_{1,2}}(U) + 4t_{K_3^+}(U) + t_{C_4}(U)\big),$$

where K_3^+ denotes the triangle plus a pendant edge. The same argument for $m_{C_4}(W)$ yields

$$m_{C_4}(W) - \frac{1}{8} = \frac{1}{8}\big(2t_{2\cdot K_2}(U) + 4t_{K_{1,2}}(U) + t_{C_4}(U)\big),$$

and thus,

$$\begin{aligned} m_D(W) - \frac{1}{16} \;&-\; c(m_{C4}(W) - \frac{1}{8}) \\ &= \frac{1}{16}\big((2-4c)t_{2\cdot K_2}(U) + (8-8c)t_{K_{1,2}}(U) + 4t_{K_3^+}(U) + (1-2c)t_{C_4}(U)\big). \end{aligned} \tag{8}$$

Recall that $U = 2W-1$ is not necessarily nonnegative, but $t_{K_{1,2}}(U)$, $t_{2\cdot K_2}(U)$, and $t_{C_4}(U)$ are always nonnegative, since $t_{K_{1,2}}(U) \geq t_{K_2}(U)^2 = t_{2\cdot K_2}(U)$ and $t_{C_4}(U) \geq t_{K_{1,2}}(U)^2$. The key inequality we shall prove is

$$|t_{K_3^+}(U)| \leq \sqrt{t_{K_{1,2}}(U)t_{C_4}(U)}. \tag{9}$$

Suppose that this is true. Then (8) gives the lower bound

$$(2-4c)t_{2\cdot K_2}(U) + (8-8c)t_{K_{1,2}}(U) - 4\sqrt{t_{K_{1,2}}(U)t_{C_4}(U)} + (1-2c)t_{C_4}(U)$$

for $16\big(m_D(W) - 1/16 - c(m_{C4}(W) - 1/8)\big)$. This is nonnegative whenever $(8-8c)(1-2c) \geq 4$ and $c \leq 1/2$. Taking $0 \leq c \leq \frac{3-\sqrt{5}}{4}$ suffices for this purpose.

It remains to prove (9). Denote $\nu(x,z) := \mathbb{E}_y\, U(x,y)U(y,z)$ and $\mu(z) := \mathbb{E}_w\, U(z,w)$. Then

$$\begin{aligned} |t_{K_3^+}(U)| &= \big|\mathbb{E}\left[U(x,y)U(y,z)U(z,x)U(z,w)\right]\big| = \big|\mathbb{E}\left[\nu(x,z)U(z,x)\mu(z)\right]\big| \\ &\leq (\mathbb{E}[\nu(x,z)^2])^{1/2}(\mathbb{E}[U(z,x)^2\mu(z)^2])^{1/2} \\ &\leq (\mathbb{E}[\nu(x,z)^2])^{1/2}(\mathbb{E}[\mu(z)^2])^{1/2} = \sqrt{t_{K_{1,2}}(U)t_{C_4}(U)}. \end{aligned}$$

□

Proof of Theorem 4.1. Recall that repeated applications of Lemma 2.2 yield

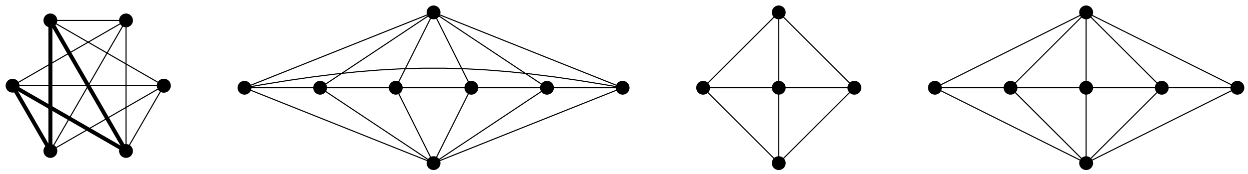
$$m_{K_{2,2,2}}(W) \geq \frac{m_D(W)^4}{m_{K_{1,2}}(W)^2 m_{C_4}(W)}.$$

By Goodman's formula (Lemma 2.1), $m_{K_{1,2}}(W) = \frac{2}{3}m_{K_3}(W) + \frac{1}{3}$. Together with the inequality $m_D(W) \geq m_{K_3}(W)^2$ that follows from Lemma 2.2 and the inequality $t_D(W) \geq t_{K_3}(W)^2/t_{K_2}(W)$, we obtain

$$m_{K_{1,2}}(W) = \frac{2}{3}m_{K_3}(W) + \frac{1}{3} \leq \frac{2}{3}\sqrt{m_D(W)} + \frac{1}{3}. \tag{10}$$

Therefore, by using Lemma 4.2,

$$m_{K_{2,2,2}}(W) \geq \frac{m_D(W)^4}{m_{K_{1,2}}(W)^2 m_{C_4}(W)} \geq \frac{c \cdot m_D(W)^4}{\left(\frac{2}{3}\sqrt{m_D(W)} + \frac{1}{3}\right)^2 \left(m_D(W) - \frac{1}{16} + \frac{c}{8}\right)}.$$



$B_4 = K_{2,2,2}$ B_6 D_2 D_4

Figure 5: Graphs B_4, B_6, D_2, and D_4.

This lower bound is a rational function h_c of $x := \sqrt{m_D(W)}$, which simplifies to

$$h_c(x) := \frac{144cx^8}{(2x+1)^2(16x^2-1+2c)}.$$

We are looking at the range $x \geq 1/4$, as $m_D(W) \geq 1/16$ by commonality of D. Taking, for example, $c = 1/7 < \frac{3-\sqrt{5}}{4}$ makes the function h_c monotone increasing on $x \geq 1/4$, and thus, $h_c(z) \geq f_c(1/4) = 2^{-11}$. This proves that $K_{2,2,2}$ is common. □

Let the k-beachball graph B_k be the graph obtained by gluing two copies of k-wheels along the k-cycle. In particular, $K_{2,2,2}$ is the 4-beachball, since it can be obtained by gluing two copies of 4-wheels along a 4-cycle. See Figure 5, where the 4-cycle is marked bold. As a straightforward generalisation of Theorem 4.1, we also prove the following theorem.

Theorem 4.3. *For every $k \geq 2$, the $2k$-beachball B_{2k} is common.*

Proof. The proof is essentially the same as Theorem 4.1 despite a slightly general setting. Let D_k be the graph obtained by adding two apex vertices to a k-edge path, i.e., it consists of k copies of diamonds glued along $K_{1,2}$'s centred at the vertices of degree three in a path-like way, as described in Figure 5. In particular, $D_1 = D$ and D_2 is the 4-wheel. Lemma 3.3 then gives

$$t_{D_k}(W) \geq \frac{t_D(W)^k}{t_{K_{1,2}}(W)^{k-1}}$$

and thus, $m_{D_k}(W) \geq m_D(W)^k / m_{K_{1,2}}(W)^{k-1}$ by Lemma 2.2.

The $2k$-beachball is then obtained by gluing two copies of D_k along the 4-cycle that contains two vertices of degree three. The standard application of the Cauchy–Schwarz inequality (or Lemma 3.3) gives $t_{B_{2k}}(W) \geq t_{D_k}(W)^2/t_{C_4}(W)$, and thus,

$$m_{B_{2k}}(W) \geq \frac{m_{D_k}(W)^2}{m_{C_4}(W)} \geq \frac{m_D(W)^{2k}}{m_{K_{1,2}}(W)^{2k-2} m_{C_4}(W)}$$

by Lemma 2.2. Then again by (10) and Lemma 4.2,

$$m_{B_{2k}}(W) \geq \frac{c \cdot m_D(W)^{2k}}{\left(\frac{2}{3}\sqrt{m_D(W)} + \frac{1}{3}\right)^{2k-2} \left(m_D(W) - \frac{1}{16} + \frac{c}{8}\right)}.$$

It remains to minimise rational function

$$h_{k,c}(x) := \frac{16 \cdot 3^{2k-2} c x^{4k}}{(2x+1)^{2k-2}(16x^2 - 1 + 2c)}$$

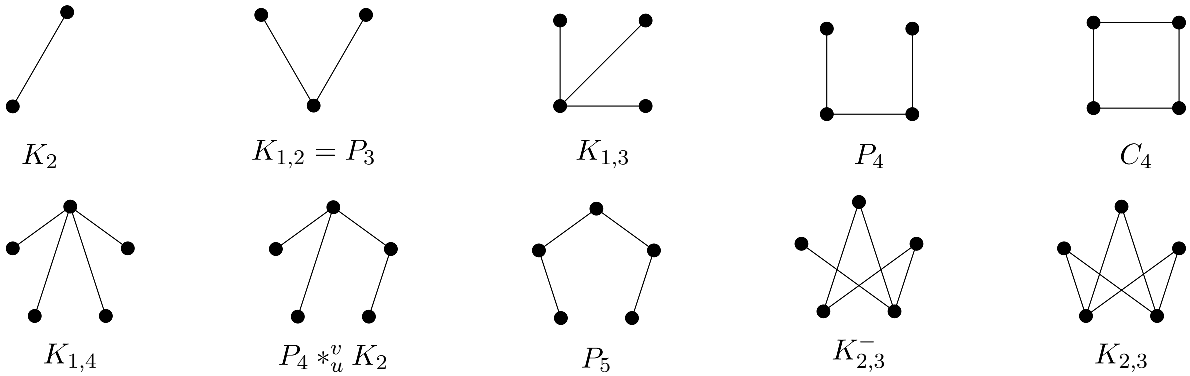


Figure 6: The ten connected bipartite graphs on at most 5 vertices from Lemma 4.4.

of $x := \sqrt{m_D(W)}$ subject to $x \geq 1/4$. Taking $c = 1/7$, $h_{k,1/7}$ is a positive constant times the function g_k, where the derivative

$$g_k'(x) = (2x+1)^{1-2k}(112x^2-5)^{-2}x^{4k-1}(112kx^3 + (112k-56)x^2 - (5k+5)x - 5k).$$

Thus, it suffices to check $p_k(x) = 112kx^3 + (112k-56)x^2 - (5k+5)x - 5k > 0$ on $x \geq 1/4$. Rearranging the terms we get $p_k(x) = 112k(x-1/4)^3 + (196k-56)(x-1/4)^2 + (72k-33)(x-1/4) + (10k-19)/4$, which is positive for $x \geq 1/4$ and $k \geq 2$. Therefore, $h_{k,1/7}(x)$ is minimised when $x = 1/4$, which implies B_{2k} is common. □

We remark that the constant $c = 1/7$ has been judiciously chosen. Indeed, if c is too large, then it gets tougher or even impossible to obtain the inequality in Lemma 4.2. Otherwise, if c is too small, then the rational function $h_c(x)$ may attain its local minimum at some $x_0 > 1/4$ and the optimisation does not work. This does happen if one tries to apply the same argument to prove that $K_{2,2,t}$ is common for $t > 2$.

However, the flag algebras allow us to prove inequalities that resemble Lemma 4.2 and can be directly applied to (7), which gives tighter bounds than the previous approach does. In particular, the following lemma generalises Lemma 4.2 for any connected bipartite graph on at most 5 vertices.

Lemma 4.4. *If H is a connected bipartite graph on at most 5 vertices and W is a graphon, then*

$$m_{H^{+1}}(W) \geq 2^{-v(H)} \cdot m_H(W). \tag{11}$$

Moreover, if $H \neq K_2$, then $m_{H^{+1}}(W) = 2^{-v(H)} \cdot m_H(W)$ if and only if $m_{C_4}(W) = 1/8$.

Proof. For any of the ten considered graphs, the proof of (11) is a straightforward flag algebra application. As the proof of (11) for $H \neq K_2$ uses $m_{C_4}(W) \geq 1/8$, the moreover part follows by complementary slackness. The flag algebra calculations certifying (11) can be downloaded from `http://lidicky.name/pub/common/`. □

Proof of Theorem 1.4. A simple convexity argument (or Lemma 3.3) gives the inequality

$$t_{H^{+a}}(W) \geq \frac{t_{H^{+1}}(W)^a}{t_H(W)^{a-1}}.$$

Lemma 2.2 then proves

$$m_{H^{+a}}(W) \geq \frac{m_{H^{+1}}(W)^a}{m_H(W)^{a-1}}$$

and by Lemma 4.4,

$$m_{H^{+a}}(W) \geq \frac{m_H(W)}{2^{a \cdot v(H)}}.$$

Since H satisfies Sidorenko's conjecture, H is common and $m_H(W) \geq 2^{1-e(H)}$. Thus, we obtain the lower bound $m_{H^{+a}}(W) \geq 2^{1-e(H)-a\cdot v(H)} = 2^{1-e(H^{+a})}$, which means that H^{+a} is common. $\square$

We remark that, if Lemma 4.4 holds for any common graph H, the proof above will immediately prove that H^{+a} is common for every positive integer a.

5 Concluding remarks

Stability. When a graph H is known to be common, it is natural to ask a stability question, i.e., whether the random colouring is (asymptotically) the unique minimiser of the number of monochromatic copies of H. In other words, is $m_H(W)$ uniquely minimised by $W = 1/2$ almost everywhere? For bipartite graphs, this question connects to the so-called *Forcing Conjecture* [26, 5] stating that if H is bipartite with at least one cycle and $p \in (0, 1)$, then $W = p$ almost everywhere uniquely minimises the number of copies of H among all graphons of density p.

For our results, one may check that that the random colouring is the unique minimiser of m_H whenever H is a triangle-tree with $\kappa(H) \geq 1$, i.e., a triangle-tree that is not a triangle-vertex-tree. Indeed, as both W and $1 - W$ must be tight for (3), inspecting the proof of [16, Theorem 2.7] yields that any minimiser of m_H must be 1/2-regular and have the 'correct' codegrees, i.e., $\int W(x, y)\mathrm{d}y = 1/2$ and $\int W(x, z)W(z, y)\mathrm{d}z = 1/4$ for almost every $x, y \in [0, 1]$, respectively. In particular, Lemma 4.2 and its applications immediately proves that $K_{1,1,2}$, $K_{1,2,2}$, and $K_{2,2,2}$ has a unique minimiser. On the other hand, there are infinitely many minimisers of m_{K_3} as any 1/2-regular graphon W has $m_{K_3} = 1/4$. This also generalises for triangle-vertex-trees and odd cycles.

In all the cases covered in Theorem 1.4 except $H = K_2$, the 'moreover' part of Lemma 4.4 yields that the random colouring is the unique minimiser. When $H = K_2$, the graph H^{+a} is simply the complete tripartite graph $K_{1,1,a}$. Therefore, the case $a = 1$ corresponds to $H^{+a} = K_3$, so every 1/2-regular graphon minimises m_{K_3}. On the other hand, if $a \geq 2$, then H^{+a} is a triangle-tree with $\kappa(H^{+a}) = a - 1$, hence by the discussion in the previous paragraph, $m_{H^{+a}}(W)$ is uniquely minimised when $W = 1/2$ almost everywhere.

Theorem 1.1 for odd cycles. It is certainly possible to generalise Theorem 1.1 by replacing triangles by odd cycles. One way is to define C_{2k+1}*-vertex-tree* and C_{2k+1}*-edge-tree* by allowing recursive additions of odd cycles along vertices or edges, respectively. For this particular class of graphs, the proof of Theorem 1.1 easily generalises. It might be possible to generalise this even further to obtain that C_{2k+1}-trees are common for every k.

Optimal pendant trees. Let H be a common graph. Then one may ask what is the smallest T that makes $T *_u^v H$ uncommon. To formalise, let

$$\mathrm{UC}(H) := \min\{e(T) : T *_u^v H \text{ is uncommon}\}.$$

Note that this parameter might not exist for some bipartite graphs H. Indeed, if H satisfies Sidorenko's conjecture, then $T *_u^v H$ satisfies the conjecture as well. In particular, H is common, and we let $\mathrm{UC}(H) = \infty$. On the other hand, if H is a triangle-edge-tree, then Lemma 3.6 and the proof of Theorem 1.2 yield a lower bound for $\mathrm{UC}(H)$ that is linear in $e(H)$. Also, Fox's result [9] implies that $\mathrm{UC}(K_{t,t,t}) = O(t^2)$, which is again linear in terms of the number of edges. It would be interesting to see more precise estimates for $\mathrm{UC}(H)$ for various non-bipartite graphs H.

Ramsey multiplicity constant of small graphs. The smallest graph whose Ramsey multiplicity constant is not known is K_4, and determining the value of $C(K_4)$ is a well-known open problem in extremal combinatorics with no conjectured value. A direct flag algebra calculation using expressions with 9-vertex subgraph densities yields $C(K_4) \geq 1/33.77 \cong 0.0296$. This is a slight improvement over previously known lower bounds [10, 27, 18], though there is a non-negligible gap from the best upper bound $1/33.0135 \cong 0.0303$ due to Thomason [28, 29].

Figure 7 describes the full list of four graphs whose commonality was questioned by Wolf in CanaDAM 2017. Indeed, Lemma 3.6 proves that H_1 and H_2 are common, and the graph H_3 was in fact proven to be common in an RSI project at MIT [19] in 2016 using flag algebras. Another flag algebra application shows that H_4 are common; in Appendix, we give a proof that both H_3 and H_4 are common. Although it is possible to fully inspect the presented proof by hand, some of the steps were obtained by using computers. It would still be interesting to find simpler 'human-friendly' proofs of the commonality of H_3 or H_4.

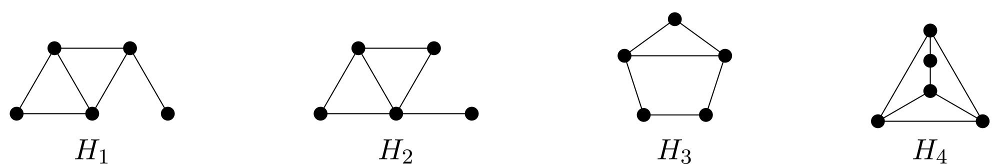


Figure 7: Wolf's list of 5-vertex graphs.

Flag algebras also prove that various 4-chromatic graphs are common: the 7-wheel and all the (connected) 7-vertex K_4-free non-3-colourable graphs are common; see Figure 8 for the complete list. We in fact suspect that all odd wheels are common, although the same approach for the 9-wheel is already beyond our computational capacity.

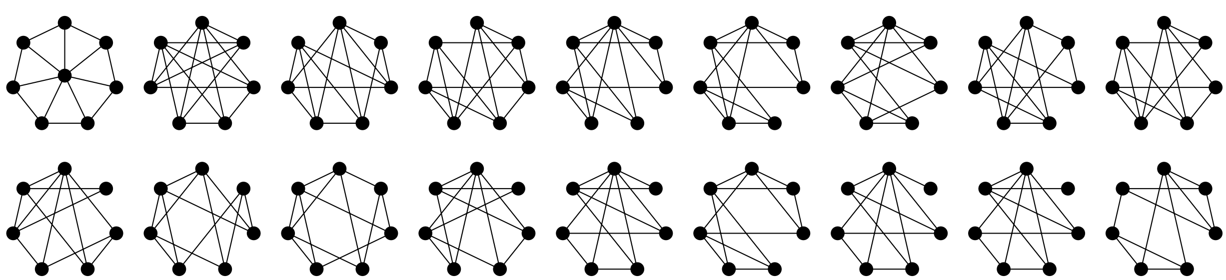


Figure 8: Non-3-colourable common graphs on 7 vertices.

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A Proof of commonality of H_3 and H_4

We present proofs of the inequalities $m_{H_3}(W) \geq 2^{-5}$ and $m_{H_4}(W) \geq 2^{-6}$ for all graphons W, where H_3 and H_4 are depicted on Figure 7. The proofs were obtained with a computer assistance using libraries CSDP [3] and QSOPT [2].

Firstly, the following three subgraph density expressions will evaluate to a nonnegative number for every graphon W due to the commonality of the corresponding graphs:

$$(1)\ 465 \cdot \left(m_{H_1}(W) - 2^{-5}\right), \qquad (2)\ 465 \cdot \left(m_{H_2}(W) - 2^{-5}\right) \quad \text{and} \quad (3)\ 48 \cdot \left(m_{C_5}(W) - 2^{-4}\right).$$

Moreover, each of these expression will be written as a linear combination of 5-vertex *induced* subgraph densities; recall that $\tau_H(W)$, the induced density of H in W, is defined as follows:

$$\tau_H(W) := \mathbb{E}\left[\prod_{ij \in E(H)} W(x_i, x_j) \prod_{ij \notin E(H)} (1 - W(x_i, x_j))\right].$$

As we aim to exploit the symmetry of the colours in Ramsey multiplicity, we let $f(\tau_H(W)) := \tau_H(W) + \tau_{\overline{H}}(W)$ for every graph H and extend f linearly to formal linear combinations of graphs.

Let $P_{ab}(W)$ be the probability measure on $[0,1]^2$ which, given a graphon W, corresponds to a uniformly sampled pair (a,b) that induces an edge. Let $T_\emptyset(W)$ and $T_{bc}(W)$ be the probability measures on $[0,1]^3$ that correspond to sampling (a,b,c) inducing an independent set and a single-edge graph $\{bc\}$, respectively. We consider the following 13 density expressions represented as sum-of-squares (we note that (6) and (7) were suggested by a computer search):

$$(4)\quad 10 \cdot f\left(\mathop{\mathbb{E}}_{T_\emptyset(W)}\left[\left(\mathbb{P}\left[x \in \bigcap_{z \in \{a,b,c\}} N_z\right] - \mathbb{P}\left[x \notin \bigcup_{z \in \{a,b,c\}} N_z\right]\right)^2\right]\right),$$

$$(5)\quad 10 \cdot f\left(\mathop{\mathbb{E}}_{T_\emptyset(W)}\left[\left(8 \cdot \mathbb{P}\left[x \notin \bigcup_{z \in \{a,b,c\}} N_z\right] - 1\right)^2\right]\right),$$

$$(6)\quad 30 \cdot f\left(\mathop{\mathbb{E}}_{T_{bc}(W)}\left[\left(2 \cdot \mathbb{P}[x \in N_b \Delta N_c] + 3 \cdot \mathbb{P}\left[x \in (N_b \Delta N_c) \setminus N_a\right]\right)^2\right]\right),$$

$$(7)\quad 30 \cdot f\left(\mathop{\mathbb{E}}_{T_{bc}(W)}\left[\left(2 \cdot \mathbb{P}[x \in N_b \Delta N_c] - 7 \cdot \mathbb{P}\left[x \in (N_b \Delta N_c) \setminus N_a\right]\right)^2\right]\right),$$

$$(8)\quad 30 \cdot f\left(\mathop{\mathbb{E}}_{T_{bc}(W)}\left[\left(\mathbb{P}\left[x \in \bigcap_{z \in \{a,b,c\}} N_z\right] - \mathbb{P}\left[x \notin \bigcup_{z \in \{a,b,c\}} N_z\right]\right)^2\right]\right),$$

$$(9)\quad 30 \cdot f\left(\mathop{\mathbb{E}}_{T_{bc}(W)}\left[\left(\mathbb{P}\left[(x \in N_a \setminus \bigcup_{z \in \{b,c\}} N_z\right] - \mathbb{P}\left[x \notin \bigcup_{z \in \{a,b,c\}} N_z\right]\right)^2\right]\right),$$

$$(10)\quad 30 \cdot f\left(\mathop{\mathbb{E}}_{T_{bc}(W)}\left[\left(\mathbb{P}\left[x \in \bigcap_{z \in \{b,c\}} N_z \setminus N_a\right] - \mathbb{P}\left[x \notin \bigcup_{z \in \{a,b,c\}} N_z\right]\right)^2\right]\right),$$

$$(11)\quad 30 \cdot f\left(\mathop{\mathbb{E}}_{T_{bc}(W)}\left[\left(\mathbb{P}\left[x \in (N_b \Delta N_c) \setminus N_a\right] - 2 \cdot \mathbb{P}\left[x \notin \bigcup_{z \in \{a,b,c\}} N_z\right]\right)^2\right]\right),$$

$$(12)\quad 30 \cdot f\left(\mathop{\mathbb{E}}_{T_{bc}(W)}\left[\left(\mathbb{P}\left[x \in (N_b \Delta N_c) \cap N_a\right] - 2 \cdot \mathbb{P}\left[x \notin \bigcup_{z \in \{a,b,c\}} N_z\right]\right)^2\right]\right),$$

$$(13)\quad 15 \cdot \mathop{\mathbb{E}}_{a \in [0,1]}\left[\left(\left(2 \cdot \mathbb{P}[x \in N_y] - 1\right) \cdot \left(\mathbb{P}\left[x \in N_a \wedge y \in N_a\right] - \mathbb{P}\left[x \notin N_a \wedge y \notin N_a\right]\right)\right)^2\right],$$

$$(14)\quad 15 \cdot f\left(\mathop{\mathbb{E}}_{P_{ab}(W)}\left[\left(\mathbb{P}\left[x \in N_a\right] - \mathbb{P}\left[x \in N_b\right]\right)^2\right]\right),$$

$$(15)\quad 15 \cdot \mathop{\mathbb{E}}_{a \in [0,1]}\left[\left(2 \cdot \mathbb{P}\left[x \in N_a \wedge y \in N_a\right] + 2 \cdot \mathbb{P}\left[x \notin N_a \wedge y \notin N_a\right] - 1\right)^2\right], \text{ and}$$

$$(16)\quad 30 \cdot f\left(\mathop{\mathbb{E}}_{P_{ab}(W)}\left[\mathbb{P}[x \in N_a \cap N_b] \cdot \left(2 \cdot \mathbb{P}\left[y \in N_a \Delta N_b\right] - 1\right)^2\right]\right),$$

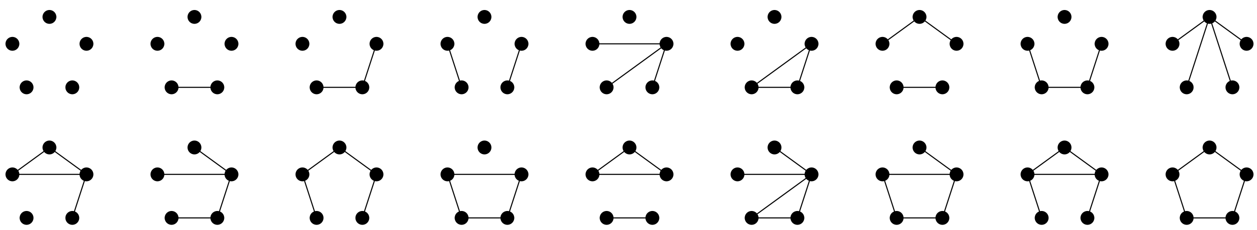

Figure 9: Non-isomorphic partitions of $E(K_5)$ into two parts represented by black and white edges.

where x and y are uniformly sampled vertices of W, and $x \in N_\star$ abbreviates the event of sampling an edge between x and $\star$. Clearly, each expression evaluates to a nonnegative number for any W and can be written as a linear combination of 5-vertex induced subgraph densities.

As there are 34 non-isomorphic 5-vertex graphs and two of them are self-complementary, there are exactly 18 non-isomorphic partitions of $E(K_5)$ into two parts (see Figure 9). Therefore, we may identify each expression described in the previous paragraph with a vector from $\mathbb{R}^{18}$ simply by letting its i-th coordinate to be the coefficient of the i-th graph in Figure 9 in the corresponding linear combination. We denote these vectors by $w_1, w_2, \ldots, w_{16}$, and let $M := (w_1|w_2|\cdots|w_{16})$ be the corresponding 18×16 matrix. Next, let v_A and v_B be the vectors from $\mathbb{R}^{18}$ representing the expressions $480 \cdot \big(m(H_3) - 2^{-5}\big)$ and $960 \cdot \big(m(H_4) - 2^{-6}\big)$, respectively. Then v_A, v_B, and M are

$$\begin{pmatrix} 465 \\ 177 \\ 33 \\ 81 \\ -15 \\ -15 \\ 17 \\ 1 \\ -15 \\ -15 \\ -15 \\ -7 \\ -15 \\ -15 \\ -15 \\ -15 \\ -15 \\ -15 \end{pmatrix}, \begin{pmatrix} 945 \\ 273 \\ 17 \\ 113 \\ -15 \\ -15 \\ 17 \\ -15 \\ -15 \\ -15 \\ -15 \\ -15 \\ -15 \\ -15 \\ -15 \\ -15 \\ -15 \\ -15 \end{pmatrix}, \text{ and } \begin{pmatrix} 465 & 465 & 45 & 10 & 490 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 15 & 0 & 15 & 0 \\ 177 & 177 & 21 & 1 & 7 & 0 & 0 & 3 & 3 & 3 & 12 & 12 & 3 & 0 & 3 & 3 \\ 49 & 65 & 5 & 0 & -27 & 0 & 0 & 1 & 0 & 0 & -8 & 0 & -3 & 1 & 3 & -3 \\ 49 & 17 & 13 & 0 & 4 & 0 & 0 & 2 & -2 & 2 & 8 & 8 & 3 & -2 & -9 & 6 \\ 9 & 9 & -3 & -1 & -28 & 75 & 12 & 0 & 0 & 0 & 3 & 0 & -3 & 3 & -9 & 6 \\ -15 & 33 & -3 & 1 & 4 & 0 & 0 & 0 & 0 & -6 & 0 & 0 & -3 & 3 & 15 & 6 \\ 1 & -15 & 5 & 0 & 2 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & -3 & -9 & 0 \\ 1 & 1 & 1 & 0 & 3 & -25 & -4 & 0 & 0 & 0 & -3 & -4 & -1 & 0 & 3 & -5 \\ -15 & -15 & -3 & -4 & -28 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 3 & 6 & -9 & 6 \\ -15 & -7 & -3 & 0 & 3 & 25 & 4 & -1 & 0 & -1 & 1 & 0 & 1 & 3 & 3 & -1 \\ -7 & -15 & -3 & 0 & 2 & 24 & 5 & 0 & 0 & 0 & 0 & -3 & -1 & 0 & -9 & 0 \\ -15 & -15 & 1 & 0 & 1 & -24 & -5 & 0 & 0 & 0 & 0 & 1 & 1 & -3 & 3 & -3 \\ -15 & -15 & -3 & 0 & 2 & -100 & -16 & 0 & 0 & 0 & 4 & 0 & -1 & 0 & 15 & 8 \\ -15 & -15 & -3 & 1 & 1 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & -3 & -3 & 3 & 9 \\ -15 & -15 & -3 & 0 & 3 & 0 & 0 & -2 & 0 & 1 & 0 & 0 & 3 & 4 & 3 & -1 \\ -15 & -15 & -3 & 0 & 1 & -16 & 45 & 0 & 0 & 0 & 0 & 1 & -1 & -2 & 3 & 1 \\ -15 & -15 & -3 & 0 & 2 & 40 & -40 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -9 & 0 \\ -15 & -15 & 5 & 0 & 0 & -40 & -250 & 0 & 0 & 0 & 0 & 10 & 5 & -5 & 15 & 10 \end{pmatrix},$$

respectively. Let M_A and M_B be the submatrices of M obtained by deleting the last and the second to last column, respectively. It follows both M_A and M_B have rank 15 and the unique x_A and x_B that satisfy $v_A = M_A x_A$ and $v_B = M_B x_B$ have nonnegative entries, explicitly given as follows:

$$x_A = \frac{1}{133168} \times \begin{pmatrix} 22852 \\ 10730 \\ 448079 \\ 6584 \\ 13776 \\ 1168352 \\ 22852 \\ 1351168 \\ 9280 \\ 513184 \\ 43384 \\ 7888 \\ 172057 \\ 329614 \\ 45472 \end{pmatrix} \qquad x_B = \frac{1}{13601} \times \begin{pmatrix} 9628 \\ 2465 \\ 19430 \\ 780 \\ 2520 \\ 56144 \\ 9628 \\ 133168 \\ 19952 \\ 19488 \\ 14268 \\ 6728 \\ 71746 \\ 14268 \\ 18676 \end{pmatrix}.$$

Thus, $m_{H_3}(W) \geq 2^{-5}$ and $m_{H_4}(W) \geq 2^{-6}$ for every graphon W.