Counterexamples to a conjecture of Harris on Hall ratio

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Abstract

The Hall ratio of a graph G is the maximum value of $v(H)/\alpha(H)$ taken over all non-null subgraphs $H \subseteq G$. For any graph, the Hall ratio is a lower-bound on its fractional chromatic number. In this note, we present various constructions of graphs whose fractional chromatic number grows much faster than their Hall ratio. This refutes a conjecture of Harris.

1 Introduction

A graph G is *k-colorable* if its vertices can be colored with k colors so that adjacent vertices receive different colors. The minimum integer k such that G is k-colorable is called *the chromatic number* of G, and it is denoted by $\chi(G)$.

Various refinements and relaxations of the chromatic number have been considered in the literature. One of the classical and most studied ones is the *fractional chromatic number*, which we denote by $\chi_f(G)$; see Section 2.1 for its definition.

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A basic averaging argument reveals that $\chi_f(G) \geq v(G)/\alpha(G)$, where $v(G)$ and $\alpha(G)$ are the number of vertices and the size of a largest independent set in G, respectively. Moreover, since $\chi_f(G) \geq \chi_f(H)$ for a subgraph $H \subseteq G$, it holds that

$$\chi_f(G) \geq \frac{v(H)}{\alpha(H)} \quad \text{for every non-null } H \subseteq G.$$

We define $\rho(G)$ — the *Hall ratio* of a graph G — to be the best lower-bound obtained in this way, i.e.,

$$\rho(G) := \max_{\emptyset \neq H \subseteq G} \frac{v(H)}{\alpha(H)}.$$

How tight is $\rho(G)$ as a lower bound for $\chi_f(G)$? In 2009, Johnson [10] suggested that there are graphs G where the value of $\chi_f(G)/\rho(G)$ is unbounded. In 2016, Harris explicitly conjectured the opposite.

Conjecture 1 ([7, Conjecture 6.2])**.** *There exists C such that $\chi_f(G) \leq C \cdot \rho(G)$ for every graph G.*

In 2016, Barnett [2] constructed graphs showing that if such a constant C exists, then $C \geq 343/282 \sim 1.216$ improving an earlier bound 1.2 [3]. Our first result refutes Conjecture 1.

Theorem 2. *There exists K_0 such that for every $K \geq K_0$, there is a graph G with $\rho(G) \leq K$ and $\chi_f(G) > K^2/33$.*

The proof of Theorem 2 is very short and simple, modulo some standard results about random graphs. The following two theorems strengthen Theorem 2 at the expense of somewhat more technical proofs.

Theorem 3. *There exists K_0 such that for every $K \geq K_0$ there is a K_5-free graph G with $\rho(G) \leq K$ and $\chi_f(G) > K^2/82$.*

Theorem 4. *There exists K_0 such that for all $K \geq K_0$ there is a graph G with $\rho(G) \leq K$ and $\chi_f(G) \geq e^{\ln^2(K)/3}$.*

This note is organized as follows. In Section 2, we recall definitions and properties of the fractional chromatic number, and Erdős-Rényi random graphs. Proofs of our results are in Section 3. We conclude the note by Section 4 with related open problems.

2 Definitions and preliminaries

The join of two graphs G_1 and G_2, which we denote by $G_1 \wedge G_2$, is obtained by taking vertex-disjoint copies of G_1 and G_2, and adding all the edges between $V(G_1)$ and $V(G_2)$. More generally, for graphs $G_1, G_2, \ldots G_\ell$, we write $\bigwedge_{i=1}^{\ell} G_i$ to denote $\left(\bigwedge_{i=1}^{\ell-1} G_i\right) \wedge G_\ell$.

For an ℓ-vertex graph H with vertices $1, \ldots, \ell$ and a collection of ℓ vertex-disjoint graphs $G_1, \ldots, G_\ell$, we define $H\{G_1, \ldots, G_\ell\}$ to be the graph obtained by taking a union $G_1, \ldots, G_\ell$, and, for every edge $ij \in E(H)$, adding all the edges between $V(G_i)$ and $V(G_j)$. Note that if $G_1 \cong \ldots \cong G_\ell$, then $H\{G_1, \ldots, G_\ell\}$ corresponds to the composition (also known as the lexicographic product) of G and H. Also, observe that

$$K_\ell\{G_1, \ldots, G_\ell\} = \bigwedge_{i=1}^{\ell} G_i.$$

2.1 Fractional chromatic number

We present a definition of the fractional chromatic number based on a linear programming relaxation of an integer program computing the ordinary chromatic number. For a graph G, let $\mathcal{I}(G)$ be the set of all (maximal) independent sets. Let FRACC be the following linear program.

$$\textsc{fracc}\begin{cases} \text{Minimize} & \displaystyle\sum_{I \in \mathcal{I}(G)} x_I \\ \text{subject to} & \displaystyle\sum_{\substack{I \in \mathcal{I}(G) \\ v \in I}} x_I \geq 1 & \text{for } v \in V(G); \\ & x_I \geq 0 & \text{for } I \in \mathcal{I}(G). \end{cases}$$

Furthermore, let FRACD be the following program, which is the dual of FRACC.

$$\textsc{fracd}\begin{cases} \text{Maximize} & \displaystyle\sum_{v \in V(G)} y(v) \\ \text{subject to} & \displaystyle\sum_{v \in I} y(v) \leq 1 & \text{for } I \in \mathcal{I}(G); \\ & y(v) \geq 0 & \text{for } v \in V(G). \end{cases}$$

Since these two linear programs are dual of each other, the LP-duality theorem ensures that they have the same value, which we denote by $\chi_f(G)$.

Let us now mention a different way to introduce the fractional chromatic number. As we have already mentioned, $\alpha(G) \geq v(G)/\chi_f(G)$. Moreover, the lower-bound stays valid even in the setting where the vertices have weights, and we measure the size of an independent set by the proportion of the weight it occupies rather than its cardinality.

More precisely, let $G = (V, E)$ be a graph and $w : V \to \mathbb{R}_+$ a weight function. Let $\alpha(G, w)$ be the maximum sum of the weights of the vertices that form an independent set, i.e.,

$$\alpha(G, w) := \max_{I \in \mathcal{I}} \sum_{v \in I} w(v)\,.$$

If we rescale an optimal solution of FRACC by a factor $1/\chi_f(G)$ and interpret it as a probability distribution on $\mathcal{I}$, the linearity of expectation yields that

$$\alpha(G,w) \geq \mathbb{E}_I\left[\sum_{v\in I} w(v)\right] = \sum_{v\in V} w(v)\cdot \sum_{\substack{I\in\mathcal{I}(G)\\ v\in I}} \frac{x_I}{\chi_f(G)} \geq \frac{\sum_{v\in V} w(v)}{\chi_f(G)}\,.$$

On the other hand, any optimal solution of FRACD yields a weight function w_0 for which the bound is tight, i.e., $\alpha(G,w_0) = \sum_{v\in V} w_0(v)/\chi_f(G)$. Therefore,

$$\chi_f(G) = \sup_{w:V\to[0,1]} \frac{\sum_{v\in V} w(v)}{\alpha(G,w)}\,.$$

Note that the Hall ratio can be viewed as an integral version of the above, since

$$\rho(G) = \max_{w:V\to\{0,1\}} \frac{\sum_{v\in V} w(v)}{\alpha(G,w)}\,.$$

For other possible definitions of the fractional chromatic number, see [14].

We finish this section with a straightforward generalization of the fact that the fractional chromatic number of the composition of two graphs is equal to the product of their fractional chromatic numbers.

Proposition 5. *Let H be a graph with the vertex-set $\{1,\dots,\ell\}$ and let $G_1,\dots,G_\ell$ be graphs. It holds that $\chi_f(H)\cdot\min_{i\in[\ell]}\chi_f(G_i) \leq \chi_f(H\{G_1,\dots,G_\ell\})$.*

Proof. Without loss of generality, we may assume that $V(G_i) = \{1,\dots,v(G_i)\}$. Let $w_1^H,\dots,w_\ell^H$ be any optimal solution of the dual program FRACD for H, and for every $i\in[\ell]$, let $w_1^i,\dots,w_{v(G_i)}^i$, be any optimal solution of FRACD for G_i.

Let $G = H\{G_1,\dots,G_\ell\}$. For a vertex $(i,j)\in V(G)$, where $i\in[\ell]$ and $j\in[v(G_i)]$, we set $y_{i,j} := w_i^H\cdot w_j^i$. It holds that

$$\sum_{(i,j)\in V(G)} y_{i,j} = \sum_{i\in[\ell]} w_i^H \cdot \sum_{j\in V(G_i)} w_j^i = \sum_{i\in[\ell]} w_i^H\cdot\chi_f(G_i) \geq \chi_f(H)\cdot\min_{i\in[\ell]}\chi_f(G_i)\,.$$

We claim that $(y_{i,j})$, where $(i,j)\in V(G)$, is a feasible solution of FRACD for G.

Indeed, fix any $I\in\mathcal{I}(G)$. For $i\in[\ell]$, let $I_i := \{j\in[v(G_i)] : (i,j)\in I\}$. Since $I_i\in\mathcal{I}(G_i)$, it holds that

$$\sum_{j\in I_i} w_i^H\cdot w_j^i = w_i^H\cdot\sum_{j\in I_i} w_j^i \leq w_i^H\,.$$

On the other hand, the set $I_H := \{i\in[\ell] : \exists(i,j)\in I\}$ is independent in H. Therefore,

$$\sum_{(i,j)\in I} y_{i,j} = \sum_{i\in I_H}\sum_{j\in I_i} y_{i,j} = \sum_{i\in I_H} w_i^H\cdot\sum_{j\in I_i} w_j^i \leq \sum_{i\in I_H} w_i^H \leq 1\,.$$

□

We note that an analogous composing of optimal solutions of FRACC yields $\chi_f(H\{G_1,\dots,G_\ell\}) \leq \chi_f(H)\cdot\max_{i\in[\ell]}\chi_f(G_i)$, but we will never need this bound.

2.2 Sparse Erdős-Rényi random graphs

Let $G_{n,p}$ be a random graph on $\{1, 2, \ldots, n\}$ where each pair of vertices forms an edge independently with probability p. We now recall some well-known properties of $G_{n,\frac{D}{n}}$ we are going to use.

Proposition 6. *There exists C_0 such that for every $C \geq C_0$ the following is true: There exists $n_0 = n_0(C) \in \mathbb{N}$ such that for every $n \geq n_0$ there is an n-vertex triangle-free graph $G = G^1(n, C)$ with the following properties:*

(A) $1.001 \cdot C > \chi(G) \geq \chi_f(G) \geq \frac{n}{\alpha(G)} > C$, *and*

(B) for all $k \leq \sqrt{\ln n}$, every k-vertex subgraph of G has at most k edges.

Proof. Suppose that C and n are sufficiently large, and let $D > 1$ be such that $C = \frac{D}{2 \cdot \ln D}$. By [6] and [12], a random graph $G_{n,\frac{D}{n}}$ satisfies with high probability $\alpha(G_{n,\frac{D}{n}}) > n/C$ and $\chi(G_{n,\frac{D}{n}}) < 1.001 \cdot C$, respectively.

Next, the expected number of subgraphs H in $G_{n,\frac{D}{n}}$ with $v(H) \leq \sqrt{\ln n}$ and more than $v(H)$ edges is at most

$$\sum_{k=3}^{\sqrt{\ln n}} 2^{k^2} \cdot n^k \cdot \left(\frac{D}{n}\right)^{k+1} \leq \sqrt{\ln n} \cdot \frac{D^{\sqrt{\ln n}+1}}{n^{1-\ln 2}} = O\left(n^{-0.3}\right).$$

Therefore, by Markov's inequality, with high probability $G_{n,\frac{D}{n}}$ has no such H.

Finally, Schürger [15] showed that the number of triangles in $G_{n,\frac{D}{n}}$ converges to the Poisson distribution with mean $\Theta(D^3)$. In particular, with a positive probability, $G_{n,\frac{D}{n}}$ is triangle-free, which finishes the proof. □

3 Counter-examples to Conjecture 1

We start with a simple construction of a sequence of graphs for which $\chi_f(G) \gg \rho(G)$. Each graph G is the join of the graphs $G^1(n_i, C)$ of very different orders.

Proof of Theorem 2. Let C_0 be the constant from Proposition 6, and $K_0 := 8C_0$.

Given $K \geq K_0$, let $\ell := \lfloor K/4 \rfloor$, $C := K/8$, and $n_1 := n_0(C)$ from Proposition 6. For all $j \in [\ell - 1]$, let $n_{j+1} := e^{2 \cdot n_j^2}$, and, for all $i \in [\ell]$, let $G_i := G^1(n_i, C)$. We set $G := \bigwedge_{i=1}^{\ell} G_i$.

By Proposition 5, $\chi_f(G) > \ell \cdot C > K^2/33$. It only remains to prove that $\rho(G) \leq K$, i.e., that $\alpha(H) \geq v(H)/K$ for every induced subgraph H of G.

Fix $X \subseteq V(G)$, and let $X_i := V(G_i) \cap X$ for $i \in [\ell]$. We split the indices into two categories, *small* and *big*, based on $|X_i|$ with respect to $v(G_i) = n_i$. Specifically, let

$$\mathcal{S} := \left\{ i \in [\ell] : |X_i| < \sqrt{\ln n_i} \right\}, \quad \text{and} \quad \mathcal{B} := [\ell] \setminus \mathcal{S}.$$

Next, let H_S and H_B be the subgraphs of $G[X]$ induced by $\bigcup_{i \in \mathcal{S}} X_i$ and $\bigcup_{i \in \mathcal{B}} X_i$, respectively, and v_s and v_b their respective orders. In both of these subgraphs, we can find quite large independent sets.

Claim 7. *H_S has an independent set of size at least $4v_s/3K$.*

Proof. Let $i \in \mathcal{S}$. By the property (B) of G_i established in Proposition 6, any subgraph of $G[X_i]$ has at most as many edges as vertices. Therefore, every connected component of $G[X_i]$ contains at most one cycle, and hence it is 3-colorable. In particular, $G[X_i]$ has an independent set of size at least $|X_i|/3$. Since $|X_j| \geq v_s/|\mathcal{S}|$ for some $j \in \mathcal{S}$, the subgraph H_S contains an independent set of size $v_s/3|\mathcal{S}| \geq v_s/3\ell \geq 4v_s/3K$. $\square$

Claim 8. *H_B has an independent set of size at least $4v_b/K$.*

Proof. Let m be the largest element of $\mathcal{B}$. Since G_m is (0.51ℓ)-colorable, $G[X_m]$ contains an independent set of size at least $1.9 \cdot |X_m|/\ell$. If $m = 1$, then $|X_m| = v_b$. On the other hand, if $m \geq 2$, then

$$1.9 \cdot |X_m| \geq |X_m| + 0.9 \cdot \sqrt{\ln n_m} > |X_m| + 1.2 \cdot n_{m-1} > |X_m| + \sum_{i=1}^{m-1} n_i \geq v_b.$$

We conclude that H_B has an independent set of size at least $v_b/\ell \geq 4v_b/K$. $\square$

If $v_s \geq 3|X|/4$, then we find an independent set of size at least $|X|/K$ in H_S by Claim 7. Otherwise, $v_b \geq |X|/4$, and Claim 8 guarantees an independent set in H_B of size at least $|X|/K$. $\square$

3.1 K_5-free and iterated constructions

As we have already noted in Section 2, the graph $G = \bigwedge_{i=1}^{\ell} G_i$ constructed in Theorem 2 can be equivalently viewed as $K_\ell\{G_1, G_2, \ldots, G_\ell\}$. An adaptation of the proof of Theorem 2 will show that replacing K_ℓ by a graph from Proposition 6 yields another graph G^2 with $\chi_f\left(G^2\right) \sim \left(\rho\left(G^2\right)\right)^2$. However, as all the graphs involved in the composition are now triangle-free, G^2 will be K_5-free.

But we do not need to stop here. Since we have now much better control on the chromatic numbers of small subgraphs in G^2 than in the original graph G, replacing the graphs $G_i = G(n_i, C)$ in the composition by n_i-vertex variants of G^2 yields a graph G^3 with $\chi_f\left(G^3\right) \sim \left(\rho\left(G^3\right)\right)^3$. Repeating this procedure k-times leads to a construction of a graph G^{k+1} with $\chi_f\left(G^{k+1}\right) \sim \left(\rho\left(G^{k+1}\right)\right)^{k+1}$.

In order to present our proofs of Theorems 3 and 4, we need to introduce some additional notation. Let us start with recalling the Knuth's up-arrow notation

$$a \uparrow^{(k)} b = \begin{cases} a^b & \text{if } k = 1, \\ 1 & \text{if } k \geq 1 \text{ and } b = 0, \\ a \uparrow^{(k-1)} \left(a \uparrow^{(k)} (b-1)\right) & \text{otherwise,} \end{cases}$$

where $a, b, k \in \mathbb{N}$, and its inverse $a \downarrow^{(k)} n$, which is the largest integer b such that $n \geq a \uparrow^{(k)} b$. Using this, we define the following Ackermann-type function $F_k(b)$ and its inverse $f_k(b)$:

$$F_k(b) := 2 \uparrow^{(k)} (b) \qquad \text{and} \qquad f_k(b) := 2 \downarrow^{(k)} (b).$$

Note that $F_1(b) = 2^b$ and $f_1(b) = \lfloor \log_2(b) \rfloor$, and for every $k \in \mathbb{N}$ it holds that $F_k(1) = 2$ and $F_k(2) = 4$. The functions also satisfy the following properties:

Fact 9. *For every $k \in \mathbb{N}$, the following holds:*

1. $f_k(f_k(F_{k+1}(n+2))) = F_{k+1}(n)$ *for every $n \in \mathbb{N}$,*
2. $f_{k+1}(4M) < f_k(f_k(M))$ *for every $M \geq F_k(F_k(7))$, and*
3. $\sum_{b=0}^{n} F_k(b) < F_k(n+1)$ *for every $n \in \mathbb{N}$.*

For a proof, see Appendix A. We are now ready to present the main lemma.

Lemma 10. *Let c_0 be the constant from Proposition 6. For every $k \in \mathbb{N}$ and $C \geq C_0$ there is $n_0 := n_0(k, C)$ such that for all $n \geq n_0$ there is an n-vertex K_{2^k+1}-free graph $G := G^k(n, C)$ with the following properties:*

- $\chi_f(G) \geq C^k$,
- $\rho(G) \leq 1.001 \cdot 3^k \cdot C$, *and*
- $G[W]$ *is 3^k-colorable for every $W \subseteq V(G)$ such that $|W| \leq f_k(f_k(n))$.*

Proof. For any fixed $C \geq C_0$, we proceed by induction on k. Since the case $k = 1$ readily follows from Proposition 6, we may assume $k \geq 2$.

Let M be the smallest positive integer such that $f_k(4M) \leq f_{k-1}(f_{k-1}(M))$. We set $n_0(k, C) := \max\{M, F_k(4 \cdot n_0(k-1, C))\}$. Given $n \geq n_0(k, C)$, we define m to be the largest integer such that

$$m + \sum_{i=2}^{m} F_k(m + 3i - 6) \leq n.$$

Note that $n < F_k(4m - 1)$, as otherwise

$$(m+1) + \sum_{i=2}^{m+1} F_k(m + 3i - 5) \leq F_k(4m - 1) \leq n.$$

Therefore, the choice of $n_0(k, C)$ ensures that $m \geq n_0(k-1, C)$.

We define $b_1 := m$, and $b_i := F_k(m + 3i - 6)$ for every $i = 2, 3, \ldots, m-1$. Finally, we set $b_m := n - \sum_{i<m} b_i$.

Let $H := G^1(m, C)$, and $G_i := G^{k-1}(b_i, C)$ for all $i \in [m]$. We define $G := H\{G_1, G_2, \ldots, G_m\}$. Clearly, the graph G contains no K_{2^k+1}. In the following three claims, we show that G has the desired three properties:

Claim 11. *$\chi_f(G) \geq C^k$.*

Proof. By the induction hypothesis, $\chi_f(H) \geq C$ and $\chi_f(G_i) \geq C^{k-1}$ for all $i \in [m]$. Therefore, Proposition 5 yields the desired lower-bound on $\chi_f(G)$. $\square$

Claim 12. *$\rho(G) \leq 1.001 \cdot 3^k \cdot C$.*

Proof. Fix an $X \subseteq V(G)$. Our aim is to show that $\alpha\left(G[X]\right) \geq |X|/\left(1.001 \cdot 3^k \cdot C\right)$. For $i \in [m]$, let X_i be $X \cap V(G_i)$. As in the proof of Theorem 2, let

$$\mathcal{S} := \left\{i \in [m] : |X_i| \leq f_{k-1}(f_{k-1}(b_i))\right\} \quad \text{and} \quad \mathcal{B} := [m] \setminus \mathcal{S}.$$

First, suppose the case $\left|\bigcup_{i \in \mathcal{S}} X_i\right| \geq |X|/3$. By the definition of $\mathcal{S}$ and the properties of G_i, every subgraph $G[X_i]$, where $i \in \mathcal{S}$, has an independent set of size at least $|X_i|/3^{k-1}$. On the other hand, $\chi(H) < 1.001 \cdot C$, so the projection of at least one of the color classes of the optimal coloring of H on $\bigcup_{i \in \mathcal{S}} X_i$ contains an independent set of size at least

$$\sum_{i \in \mathcal{S}} \frac{|X_i|}{3^{k-1}} \cdot \frac{1}{1.001 \cdot C} \geq \frac{|X|}{1.001 \cdot 3^k \cdot C}.$$

Now suppose $\left|\bigcup_{i \in \mathcal{B}} X_i\right| \geq 2|X|/3$, and let z be the maximum index in $\mathcal{B}$. If $z = 1$, then $|X_1| \geq 2|X|/3$. On the other hand, if $z \geq 2$, then

$$f_{k-1}(f_{k-1}(b_z)) \geq f_{k-1}(f_{k-1}(F_k(m + 3z - 6))) = F_k(m + 3z - 8) \geq \sum_{i<z} b_i,$$

and $|X_z| \geq |X|/3$. Since $\rho(G_z) \leq 1.001 \cdot 3^{k-1} \cdot C$ by the induction hypothesis, the subgraph $G_z[X_z]$ contains an independent set of the sought size. $\square$

Claim 13. *$G[W]$ is 3^k-colorable for every $W \subseteq V$ with $|W| \leq f_k(f_k(n))$.*

Fix a set $W \subseteq V$ of size at most $f_k(f_k(n))$. Firstly, let $Z := \{i : W \cap V(G_i) \neq \emptyset\}$. Clearly, $|Z| \leq |W| \leq f_k(f_k(n))$. Since $f_k(n) \leq 4m$ and $f_k(x) \ll \log_2 \log_2(x/4)$, we conclude that $|Z| \leq \log_2 \log_2(v(H))$. Therefore, there exists a proper 3-coloring of the induced subgraph $H[Z]$. Similarly,

$$|V(G_i) \cap W| \leq |W| \leq f_k(4m) \leq f_{k-1}(f_{k-1}(m)) \leq f_{k-1}(f_{k-1}(b_i))$$

for all $i \in [m]$. Therefore, the induction hypothesis yields that each $V(G_i) \cap W$ induces a 3^{k-1}-colorable subgraph of G, and the Cartesian product of their colorings with the 3-coloring of $H[Z]$ yields a proper 3^k-coloring of $G[W]$. $\square$

Theorem 3 is a direct consequence of Lemma 10 applied with $k = 2$. It remains to establish Theorem 4:

Proof of Theorem 4. Let $K_0 := (2C_0)^2$. Given $K \geq K_0$, let $C := \sqrt{K/1.001}$ and $k := \lfloor \log_3 C \rfloor$. Applying Lemma 10 with k and C yields an $n_0(k, C)$-vertex graph G with $\rho(G) \leq K$ and

$$\chi_f(G) \geq C^{\lfloor \log_3 C \rfloor} > e^{0.9 \cdot \ln^2(C)} > e^{\ln^2(K)/3}.$$

$\square$

4 Concluding remarks

We presented various constructions of graphs where the fractional chromatic number grows much faster than the Hall ratio, which refuted Conjecture 1. It is natural to ask whether the conclusion in Conjecture 1 can be relaxed and the fractional chromatic number of a graph is always upper-bounded by some function of its Hall ratio.

Question 14. *Is there a function $g : \mathbb{R} \to \mathbb{R}$ such that $\chi_f(G) \leq g(\rho(G))$ for every graph G?*

Theorem 4 shows that if such a function g exists, then $g(x) \geq e^{\ln^2(x)/3}$. While preparing our manuscript, we have learned that Dvořák, Ossona de Mendez and Wu [4] constructed graphs with Hall ratio at most 18 and arbitrary large fractional chromatic number. Therefore, the answer to Question 14 is no.

Conjecture 1 was partially motivated by another conjecture of Harris concerned with fractional colorings of triangle-free graphs, which was inspired by a famous result of Johansson [9] (for a recent short proof, see [13]) stating that $\chi(G) = O(\Delta/\ln\Delta)$ for every triangle-free graph G with maximum degree Δ.

Conjecture 15 ([7, Conjecture 6.4])**.** *There is C such that $\chi_f(G) \leq C \cdot d/\ln d$ for every triangle-free d-degenerate graph G.*

A classical result of Ajtai, Komlós, and Szemerédi [1] together with an averaging argument yield that $\rho(G) = O(d/\ln d)$ for G and d as above. Therefore, if Conjecture 1 could be recovered in the triangle-free setting, it would immediately yield the sought bound on χ_f in Conjecture 15.

Question 16. *Is there C such that $\chi_f(G) \leq C \cdot \rho(G)$ for every triangle-free graph G?*

In [10], it has been mentioned that the sequence of Mycelski graphs might provide a negative answer to Question 16, but we still do not know. For K_5-free graphs, Theorem 3 shows that the answer is definitely negative. As a possibly simpler question, does the answer stay negative in case of K_4-free graphs?

Question 17. *Is there C such that $\chi_f(G) \leq C \cdot \rho(G)$ for every K_4-free graph G?*

Let us conclude with an additional motivation for studying Conjecture 15. Very recently, Esperet, Kang and Thomassé [5] conjectured that dense triangle-free graphs must contain dense induced bipartite subgraphs.

Conjecture 18 ([5, Conjecture 1.5])**.** *There exists $C > 0$ such that any triangle-free graph with minimum degree at least d contains an induced bipartite subgraph of minimum degree at least $C \cdot \ln d$.*

Erdős-Rényi random graphs of the appropriate density show that the bound would be, up to the constant C, best possible. A relation between the fractional chromatic number and induced bipartite subgraphs proven in [5, Theorem 3.1] shows that if Conjecture 15 holds, then Conjecture 18 holds as well. Very recently, Kwan, Letzter, Sudakov and Tran [11] proved a slightly weaker version of Conjecture 18 where the bound $C \cdot \ln n$ is replaced by $C \cdot \ln n/\ln\ln n$.

Acknowledgments

The authors thank Zdeněk Dvořák for sharing the manuscript of [4] with them, and Józsi Balogh, Sasha Kostochka and Yani Pehova for fruitful discussions at the beginning of the project.

This work was partially supported by NSF-DMS grant #1604458 "Collaborative Research: Rocky Mountain Great Plains Graduate Research Workshops in Combinatorics".

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A Proof of Fact 9

The definitions of f_k and F_{k+1} readily yield that $f_k\left(F_{k+1}(n+1)\right) = F_{k+1}(n)$. Therefore, $f_k\left(f_k\left(F_{k+1}(n+2)\right)\right) = F_{k+1}(n)$ proving the first property.

For every $k, n \in \mathbb{N}$, a straightforward induction yields that $F_k(n) \geq n+1$. This in turn implies that $F_{k+1}(n) = F_k(F_{k+1}(n-1)) \geq F_k(n) \geq 2^n$. Similarly, for all $k \in \mathbb{N}$, the functions $F_k(\cdot)$ and $f_k(\cdot)$ are monotone non-decreasing. Therefore, for all $k \in \mathbb{N}$ and $n \geq 7$, it holds that

$$F_{k+1}(n) = F_k(F_k(F_k(F_k(F_{k+1}(n-4))))) \geq 2^{F_k\left(F_k\left(2^{n-4}\right)+1\right)} \geq 4 \cdot F_k(F_k\,(n+1)+1).$$

Since $F_k(f_k(M)+1) > M \geq F_k(f_k(M))$, we assert that $f_{k+1}(4M) < f_k(f_k(M))$ for all $M \geq F_k(F_k(7))$. Indeed, as otherwise

$$4M \geq F_{k+1}(f_{k+1}(4M)) \geq F_{k+1}(f_k(f_k(M))) \geq 4 \cdot F_k(F_k\,(f_k(f_k(M))+1)+1) > 4M,$$

a contradiction. This concludes the proof of the second property.

The last property is proven by induction on k. Indeed, the case $k = 1$ is the sum of a geometric progression. If $k \geq 2$, then by induction hypothesis

$$\sum_{b=0}^{n} F_{k+1}(b) = \sum_{b=0}^{n} F_k(F_{k+1}(b-1)) \leq \sum_{i=0}^{F_{k+1}(n-1)} F_k(i) < F_k(F_{k+1}(n-1)+1).$$

However, the right-hand side is at most $F_k(F_k(F_{k+1}(n-1))) = F_{k+1}(n+1)$.