Closing in on Hill's conjecture

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Abstract

Borrowing László Székely's lively expression, we show that Hill's conjecture is "asymptotically at least 98.5% true". This long-standing conjecture states that the crossing number $\mathrm{cr}(K_n)$ of the complete graph K_n is $H(n) := \frac{1}{4}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor$, for all $n \geq 3$. This has been verified only for $n \leq 12$. Using flag algebras, Norin and Zwols obtained the best known asymptotic lower bound for the crossing number of complete bipartite graphs, from which it follows that for every sufficiently large n, $\mathrm{cr}(K_n) > 0.905\,H(n)$. Also using flag algebras, we prove that asymptotically $\mathrm{cr}(K_n)$ is at least $0.985\,H(n)$. We also show that the spherical geodesic crossing number of K_n is asymptotically at least $0.996\,H(n)$.

1 Introduction

A long standing open problem in topological graph theory is to determine the crossing number of the complete graph K_n. We recall that the *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum number of pairwise crossings of edges in a drawing of G in the plane.

1.1 Our main results

As narrated in the illustrative survey by Beineke and Wilson [14], the problem of estimating the crossing number of complete graphs seems to have been first explored by the British artist Anthony Hill in the late 1950s. Hill found a construction that yields a drawing of K_n with exactly $\frac{1}{4}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor$ crossings, for every integer $n \geq 3$ [24]. In that paper, the following conjecture was put forward:

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Conjecture. (Hill's conjecture)

$$\mathrm{cr}(K_n) = H(n) := \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor.$$

As we recall below in our discussion of previous work, Hill's conjecture has been only verified for $n \leq 12$, and it follows from work by Norin and Zwols [34] that $\lim_{n\to\infty} \mathrm{cr}(K_n)/H(n) > 0.905$. Our main result in this paper is the following.

Theorem 1.

$$\lim_{n\to\infty} \frac{\mathrm{cr}(K_n)}{H(n)} > 0.98559895.$$

We also investigate spherical drawings of K_n. We recall that in a *spherical geodesic* drawing of a graph, the host surface is the sphere, and each edge is a minimum distance geodesic arc joining its endpoints. The *spherical geodesic crossing number* $\mathrm{cr}_{S^2}(G)$ of a graph G is the minimum number of crossings in a spherical geodesic drawing of G. This crossing number variant is of interest not only naturally in its own, but also by its connection, unveiled by Wagner [44], to the Spherical Generalized Upper Bound Conjecture.

We note that Hill's conjecture also applies to spherical geodesic drawings, since Hill's construction can be realized as a spherical geodesic drawing. Using analogous techniques as in the proof of Theorem 1, we show the following.

Theorem 2.

$$\lim_{n\to\infty} \frac{\mathrm{cr}_{S^2}(K_n)}{H(n)} > 0.99635588.$$

Actually we prove this last bound not only for spherical geodesic drawings, but for the more general class of *convex* drawings [7, 8]. A drawing D of K_n in the sphere is *convex* if, for every 3-cycle C, there is a closed disc Δ bounded by C with the following property: for any two vertices u, v contained in Δ, the edge uv is contained in Δ. We prove that the bound in Theorem 2 holds for convex drawings. Thus in particular it holds for spherical geodesic drawings, as it is easy to see that these drawings are convex.

1.2 Previous work on Hill's conjecture

We are aware of three distinct constructions that yield drawings of K_n with exactly $H(n)$ crossings. Hill's construction [24] produces *cylindrical* drawings, which are drawings in which the vertices are drawn on two concentric circles, and no edge intersects any of these circles, except at its endpoints. Blažek and Koman's construction [15] yields 2-*page drawings* of K_n, that is, drawings in which every vertex lies on the x-axis, and each edge lies (except for its endpoints) either in the upper or in the lower halfplane. Very recently, Ábrego, Aichholzer, Fernández-Merchant, Ramos, and Vogtenhuber [6] described a variant of Hill's construction that yields drawings of K_n with $H(n)$ crossings, for every odd $n \geq 11$.

Hill's conjecture has been verified both for 2-page [4] and for cylindrical [5] drawings. It is also known that the conjecture holds for *monotone* drawings, that is, drawings in which each edge is drawn as an x-monotone curve [3, 11]. The new construction in [6] yields drawings that are neither 2-page nor cylindrical, but they satisfy a property called bishellability. In [2], it was proved that Hill's conjecture holds for bishellable drawings. This last result implies Hill's conjecture for 2-page, cylindrical, and monotone drawings, as all these classes of drawings are bishellable.

A straightforward counting argument shows that if Hill's conjecture holds for some odd n, then it also holds for $n+1$. In its full generality (that is, not for specific classes of drawings), the conjecture has only been verified for $n \leq 12$. For $n \leq 10$ this appears to have been reported first in [23]; recently, McQuillan and Richter [32] gave a computer-free verification of Hill's conjecture for $n = 9$ (and, by the previous observation, for $n = 10$). Pan and Richter [36] gave a computer-assisted proof for $n = 11$ (and hence for $n = 12$). Hill's conjecture for $n \leq 12$ has also been verified in [1]. This last computer-assisted verification was done under the setting of rotation systems, a framework on which we also heavily rely in this work.

The conjecture for $n = 13$ states that $\operatorname{cr}(K_{13}) = 225$. An elementary counting using $\operatorname{cr}(K_{11}) = H(11) = 100$ shows that $\operatorname{cr}(K_{13}) \geq 217$. McQuillan, Pan, and Richter [30] have ruled out the possibility that $\operatorname{cr}(K_{13}) = 217$, and since $\operatorname{cr}(K_{13})$ is an odd number [31], it follows that $\operatorname{cr}(K_{13}) \in \{219, 221, 223, 225\}$. This was further narrowed in [1], finding that $\operatorname{cr}(K_{13}) \in \{223, 225\}$.

An elementary counting using that $\operatorname{cr}(K_{13}) \geq 223$ shows that $\operatorname{cr}(K_n) \geq \frac{223}{17160} n(n-1)(n-2)(n-3) > (0.8317 + o(1))\, H(n)$. However, the best general lower bounds known for $\operatorname{cr}(K_n)$ are obtained by exploiting the close relationship between the crossing numbers of complete and complete bipartite graphs.

Recall that Zarankiewicz's conjecture states that $\operatorname{cr}(K_{p,q}) := Z(p,q) := \left\lfloor \frac{p}{2} \right\rfloor \left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{q}{2} \right\rfloor \left\lfloor \frac{q-1}{2} \right\rfloor$, for all positive integers p, q [14, 22, 43]. It follows from a result in [41] that

$$L_1 := \lim_{n \to \infty} \frac{\operatorname{cr}(K_{n,n})}{Z(n,n)} \quad \text{and} \quad L_2 := \lim_{n \to \infty} \frac{\operatorname{cr}(K_n)}{H(n)} \tag{1}$$

both exist, and that $L_2 \geq L_1$.

A counting argument using that $\operatorname{cr}(K_{5,n}) = Z(5,n)$ [26] implies that $L_1 \geq 0.8$. De Klerk, Maharry, Pasechnik, Richter, and Salazar [17] used semidefinite programming (SDP) techniques to give a lower bound on $\operatorname{cr}(K_{7,n})$, from which it follows that $L_1 > 0.83$. De Klerk, Pasechnik, and Schrijver [18] also used SDP to give a lower bound on $\operatorname{cr}(K_{9,n})$, and from this bound it follows that $L_1 > 0.859$. We also note that for each fixed integer $m \geq 3$, it is a finite problem to decide whether or not Zarankiewicz's conjecture holds for $K_{m,n}$, for every $n \geq m$ [16].

Norin and Zwols (unpublished; see [34]) used flag algebras to show that $L_1 > 0.905$. By (1) this implies that $\lim_{n\to\infty} \operatorname{cr}(K_n)/H(n) > 0.905$. Prior to our work, this was the best asymptotic lower bound known for $\operatorname{cr}(K_n)$.

For a thorough recent survey of Zarankiewicz's and Hill's conjectures, we refer the reader to [42].

We finish this survey of previous results with a few words on the spherical geodesic crossing number. This notion was introduced by Moon [33], who proved the intriguing result that if one takes a random spherical drawing of K_n (n points are placed randomly in the sphere, and each pair of points is joined by a shortest geodesic arc), then the expected number of crossings, divided by $H(n)$, is asymptotically 1. As far as we know, the best lower bound previously known for $\mathrm{cr}_{S^2}(K_n)$ is the same (asymptotically at least 0.905) as for $\mathrm{cr}(K_n)$.

1.3 An overview of our strategy

Our proof makes essential use of flag algebras. This powerful tool, introduced by Razborov [38], has been the basis of several recent groundbreaking results in a variety of combinatorial and geometric problems, such as [10, 12, 13, 19, 25, 27, 37, 39], to name just a few.

Although developed in a more general setting, flag algebras in particular provide a formalism to tackle combinatorial problems of an extremal nature, in which a result of an asymptotic nature is seeked. Using flag algebras, one can find asymptotic estimates on the size of combinatorial objects, given some information on the structure of these objects for small size instances.

In a nutshell, to prove Theorem 1 we exploit the fact that we have a complete understanding of all the good drawings of K_7 [1], and thus of their rotation systems. (In Section 2.1 we review the notions of a good drawing and of a rotation system). With this information, using flag algebras we show that out of the $\binom{n}{4}$ drawings of K_4 induced from a good drawing D of K_n (for every n sufficiently large), less than (roughly) $0.6305\binom{n}{4}$ can have 0 crossings. Therefore D must have more than $(1 - 0.6305)\binom{n}{4} = 0.3695\binom{n}{4}$ crossings, and thus $\mathrm{cr}(K_n) > 0.3695\binom{n}{4}$. Theorem 1 is just an equivalent way of writing this last inequality, using a more precise rounding of the actually computed bounds.

For Theorem 2 we proceed in an analogous manner. For this case, we use that we have the full catalogue of rotation systems that are induced from convex drawings of K_8. We obtain that out of the $\binom{n}{4}$ drawings of K_4 induced from a convex drawing of K_n, less than (roughly) $0.6272\binom{n}{4}$ can have 0 crossings.

A more detailed outline of our arguments is given in Section 2, where besides reviewing the concepts of good drawings and rotation systems, we introduce the notion of density, which plays a fundamental role in the theory of flag algebras. In Section 3 we state Theorems 3 and 4, two results in the language of flag algebras, and show that Theorems 1 and 2, respectively, follow as easy consequences. The rest of the paper is then devoted to the proof of Theorems 3 and 4.

2 Densities and rotation systems

In this section we introduce the concepts of rotation systems and densities, which are central to the proofs of Theorems 1 and 2. We will motivate the introduction of these notions by explaining their roles in the proof.

2.1 Densities in drawings of K_n

We start by recalling that a drawing of a graph is *good* if (i) no two adjacent edges intersect, other at their common endvertex; (ii) no two edges intersect each other more than once; and (iii) every intersection of two nonadjacent edges is a crossing, rather than tangential.

It is easy to show that every crossing-minimal drawing of a graph is necessarily good. Since we will only deal with crossing-minimal drawings (and with their induced subdrawings), we will assume throughout this work that all drawings under consideration are good.

In our context, we aim to find an asymptotic lower bound for $\mathrm{cr}(K_n)$. It is easy to show that if D is a good drawing of K_n, then each of the $\binom{n}{4}$ drawings of K_4 induced by D has exactly 0 or 1 crossings. Each crossing appears in exactly one such K_4, so our aim can be stated equivalently as follows: find an asymptotic *upper* bound for the proportion of non-crossing K_4s in a drawing of K_n.

Formally, for a drawing D of K_n let $d(\triangle; D)$ denote the probability that if we choose 4 vertices at random from D, the corresponding drawing of K_4 induced from D by these 4 vertices has 0 crossings. Letting $\mathrm{cr}(D)$ denote the number of crossings in D, the above definition then implies that $\mathrm{cr}(D) = \big(1 - d(\triangle; D)\big)\binom{n}{4}$. The notation $\triangle$ hints to the unique (up to isomorphism) drawing of K_4 with 0 crossings (see left hand side of Figure 1).

Thus $0 \le d(\triangle; D) \le 1$ for any drawing D of K_n with $n \ge 4$. Since K_5 cannot be drawn without crossings, it follows that $d(\triangle; D) < 1$ if D is a drawing of K_n with $n = 5$ (and, actually, for any integer $n \ge 5$).

An asymptotic reading of Hill's conjecture is that $\mathrm{cr}(K_n) = (3/8)\binom{n}{4} + O(n^3)$, and so this conjecture predicts that $d(\triangle; D)$ is asymptotically at most $(1 - 3/8) = 0.625$. What we establish in this paper is that $d(\triangle; D)$ is asymptotically less than (roughly) 0.6305. Consequently, $\mathrm{cr}(K_n)/\binom{n}{4}$ is asymptotically greater than $1 - 0.6305 = 0.3695$. An equivalent way to say this, as stated in Theorem 1, is that $\mathrm{cr}(K_n)/H(n)$ is greater than $0.3695/0.375 > 0.985$.

Our approach consists of estimating $d(\triangle; D)$, where D is a crossing-minimal drawing of K_n for some large integer n, by exploiting our complete knowledge of all good drawings of K_n for small values of n, and in particular for $n = 7$ and $n = 8$.

With Theorem 1 in mind, suppose for a moment that we limit ourselves to using the information that $\mathrm{cr}(K_7) = 9$. From this we obtain that for every drawing D_7 of K_7 we have $d(\triangle; D_7) \le \alpha := (1 - 9/\binom{7}{4}) \approx 0.742$. This readily implies that $d(\triangle; D) \le \alpha$ for every drawing D of K_n with $n \ge 7$. If there existed arbitrarily large such drawings D with $d(\triangle; D) = \alpha$, this would mean that each induced subdrawing of K_7 is crossing-minimal.

This is already impossible for $n = 8$: there are no drawings of K_8 in which each induced subdrawing of K_7 has exactly 9 crossings. Loosely speaking, it is not possible to "pack" 8 crossing-minimal drawings of K_7 into a drawing of K_8. Our approach to get the much better estimate $d(\triangle; D) < 0.6305$ (for large n) is to take the full catalogue of *all* the good drawings of K_7, and use flag algebras to investigate how these can be packed into a good drawing of K_n, for large n.

2.2 Rotation systems

To achieve this last goal, we start by turning the topological problem at hand into a combinatorial one. Instead of considering directly drawings of complete graphs, we work with *rotation systems*. A rotation system combinatorially encodes valuable information of a drawing, by recording, for each vertex v, the cyclic order in which the edges incident with v leave v (see Figure 1). Thus the rotation system of a drawing of K_n is a collection of n cyclic permutations. In general, an *abstract rotation system* [28] on a set S of n elements is a collection of n cyclic permutations, where each element $s \in S$ has an assigned cyclic permutation of the other $n-1$ elements, the *rotation* at s. We often use $s{:}s_1 s_2 \ldots s_{n-1}$ to denote that the cyclic permutation assigned to s is $s_1 s_2 \ldots s_{n-1}$. We say that S is the *ground set* of the abstract rotation system.

Throughout this work, for brevity, we shall refer to an abstract rotation system simply as a rotation system.

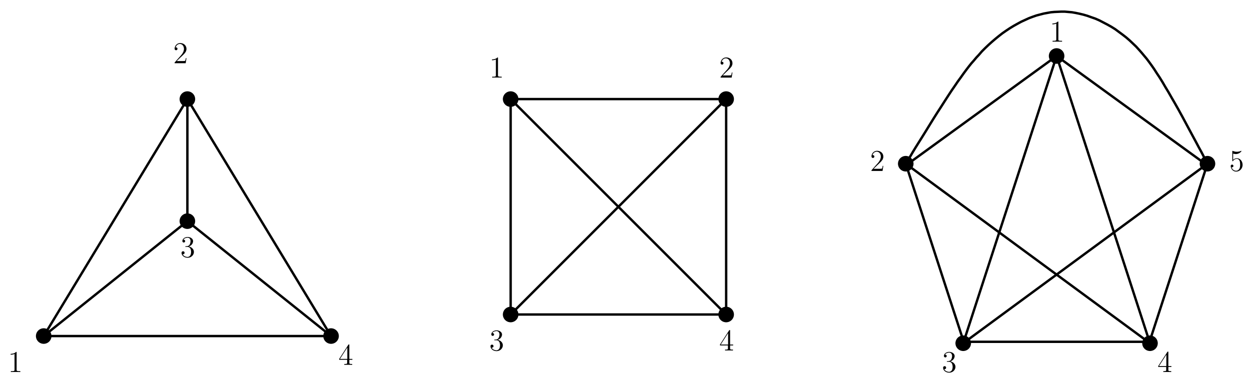

Figure 1: The left hand side drawing of K_4 induces the rotation system $N_4 := \{1{:}234, 2{:}134, 3{:}124, 4{:}132\}$. The drawing of K_4 in the center induces the rotation system $\{1{:}243, 2{:}143, 3{:}124, 4{:}123\}$. The drawing D_3 of K_5 on the right hand side induces the rotation system $\{1{:}2543, 2{:}1435, 3{:}1542, 4{:}1532, 5{:}1243\}$. We remark that since the rotation at each vertex is a *cyclic* permutation of the other vertices, we may alternatively write this last rotation system, for instance, as $\{1{:}3254, 2{:}3514, 3{:}1542, 4{:}2153, 5{:}3124\}$.

Two rotation systems are *isomorphic* if each of them can be obtained from the other simply by a relabelling of its elements. An abstract rotation system is *realizable* (respectively, *convex*) if it is isomorphic to the rotation system induced by a good drawing of K_n (respectively, by a convex drawing of K_n). Every convex rotation system is realizable, as the set of convex drawings is a (proper) subset of the collection of good drawings.

Given a rotation system R on a set S of n elements, and a subset S' of S, R naturally induces a rotation system (a *rotation subsystem*) on S', simply by removing from R all the appearences of the elements in $S \setminus S'$. For instance, if R is the rotation system

$\{1{:}234, 2{:}143, 3{:}142, 4{:}123\}$ on $S = \{1, 2, 3, 4\}$, and we let $S' = \{1, 2, 4\}$, then the rotation system on S' induced by R is $R' = \{1{:}24, 2{:}14, 4{:}12\}$.

2.3 Densities in rotation systems

The notion of density of ◬ in a drawing of K_n gets naturally extended to rotations. In general, if R, R' are rotation systems, then we let $d(R'; R)$ denote the probability that a randomly chosen rotation system of R with $|R'|$ elements is isomorphic to R'. Note that if $|R'| > |R|$, then $d(R'; R) = 0$.

There is (up to isomorphism) a unique rotation system N_4 on 4 elements induced by a drawing of K_4 with no crossings; again we refer the reader to Figure 1, in whose caption N_4 is presented.

For a (realizable or not) rotation system R on $n \geq 4$ elements, let $d(N_4; R)$ denote the probability that a randomly chosen rotation subsystem of R with 4 elements is isomorphic to N_4. Clearly, if R is realized by a drawing D of K_n, then $d(◬; D) = d(N_4; R)$. Thus, in order to prove Theorem 1, it suffices to show that $d(N_4; R) < 0.6305$ for every sufficiently large realizable rotation system R. For Theorem 2, we show that the bound $d(N_4; R) < 0.6272$ holds if R is convex.

We know the family $\mathcal{E}_7$ of 22,730 non-isomorphic realizable rotation systems on 7 elements (this is discussed in Section 4). A trivial, but key observation, is that if R is a realizable rotation system on $n \geq 7$ elements, then each of the rotation subsystems of R on 7 elements is (isomorphic to a rotation) in $\mathcal{E}_7$.

What we show is that *if R is a realizable rotation system on n elements such that each of its rotation subsystems on 7 elements is in $\mathcal{E}_7$, then* $d(N_4; R) < 0.6305$ (as long as R is sufficiently large). We show this by using tools from flag algebras. The size 22,730 turns out to be small enough to be handled with these techniques.

For Theorem 2 we proceed in a similar way. The improvement over the general bound in Theorem 1 is obtained using the set $\mathcal{C}_8$ of convex realizable systems, which is also small enough (7,360 rotations) to use the flag algebras approach.

3 Convergent subsequences of rotation systems: proof of Theorems 1 and 2

In this section we show that Theorems 1 and 2 follow from two results on sequences of rotation systems. These statements involve the notion of convergence, from the flag algebras framework.

Let $R_1, R_2, \ldots,$ be an infinite sequence of rotation systems, where $|R_i| < |R_{i+1}|$ for $i = 1, 2, \ldots$. The sequence $R_1, R_2, \ldots$ is *convergent* if, for each fixed rotation system R', the sequence $\{d(R'; R_i)\}_{i=1}^{\infty}$ converges.

A standard compactness argument, using Tychonoff's theorem, shows that every infinite sequence of rotation systems has a convergent subsequence. In particular, there exist convergent sequences of realizable, and of convex, rotation systems. Such convergent sequences are the central objects in the next statements which, as we shall see shortly, easily imply Theorems 1 and 2, respectively.

Theorem 3. *Let $R_1, R_2, \ldots$ be a convergent sequence of realizable rotation systems. Then*

$$\lim_{i\to\infty} d(N_4; R_i) < A := \frac{22064013752809590266065131421016}{35000000000000000000000000000000} < 0.630400393.$$

Theorem 4. *Let $R_1, R_2, \ldots$ be a convergent sequence of convex rotation systems. Then*

$$\lim_{i\to\infty} d(N_4; R_i) < B := \frac{43909978466574504806937629255000}{70000000000000000000000000000000} < 0.627285407.$$

The rest of this paper will be devoted to the proofs of these statements. We close this section by showing how Theorem 1 follows from Theorem 3. The proof that Theorem 2 follows from Theorem 4 is totally analogous.

Proof of Theorem 1, assuming Theorem 3. Let $D_1, D_2, \ldots$ be an infinite sequence of drawings such that, for $i \in \mathbb{N}$, D_i is a crossing-minimal drawing of K_i. For $i \in \mathbb{N}$, let R_i be the rotation system induced by D_i.

A well-known argument using Tychonoff's theorem shows that $R_1, R_2, \ldots$ has a convergent subsequence $R_{n(1)}, R_{n(2)}, \ldots$. Since (as observed in Section 2.3) $d(N_4; R_{n(i)}) = d(\triangle; D_{n(i)})$ for $i = 1, 2, \ldots$, from Theorem 3 we have $\lim_{i\to\infty} d(\triangle; D_{n(i)}) < A$.

The crossing-minimality of each $D_{n(i)}$ means that $\mathrm{cr}(K_{n(i)}) = \mathrm{cr}(D_{n(i)})$ for $i \in \mathbb{N}$. Now since $\mathrm{cr}(D_{n(i)}) = \bigl(1 - d(\triangle; D_{n(i)})\bigr)\binom{n}{4}$ for each $i \in \mathbb{N}$, the convergence of $d(\triangle; D_{n(1)})$, $d(\triangle; D_{n(2)}), \ldots$ to a number smaller than A implies that

$$\frac{\mathrm{cr}(K_{n(1)})}{\binom{n(1)}{4}}, \frac{\mathrm{cr}(K_{n(2)})}{\binom{n(2)}{4}}, \ldots \tag{2}$$

is a convergent sequence, whose limit is greater than $1 - A$.

Since the sequence in (2) is a subsequence of the sequence $\{\mathrm{cr}(K_n)/\binom{n}{4}\}_{n=1}^{\infty}$, and this sequence is also convergent [41], then $\lim_{n\to\infty} \mathrm{cr}(K_n)/\binom{n}{4} > 1 - A$. Since $\lim_{n\to\infty} H(n)/\binom{n}{4} = 3/8 = 0.375$, then $\lim_{n\to\infty} \mathrm{cr}(K_n)/H(n) > (1 - A)/0.375 > 0.98559895$. □

4 Small rotation systems

As described in Section 2, an essential ingredient in the proof of Theorem 3 is that we know the full collection of all non-isomorphic realizable rotation systems on 7 elements. Analogously, to prove Theorem 4 we use the collection of all non-isomorphic convex rotation systems on 8 elements.

In this section we describe how these families are obtained.

4.1 Realizable rotation systems on 7 elements

For each integer $n \geq 3$, we use $\mathcal{E}_n$ to denote the set of all non-isomorphic realizable rotation systems on n elements.

Aichholzer and Pammer wrote code to obtain all non-isomorphic realizable rotation systems on n elements for $n \leq 9$, with the results reported in [1, Table 1] (see also [35]). We note that in [1] a distinct notion of isomorphism (to the one used in this paper) is used. Let us say that two rotation systems R, R' are *equivalent* if either R is isomorphic to R', or if R' is isomorphic to the system obtained by taking the inverse of each of the rotations in R (that is, if R' is the *inverse* R^{-1} of R). Under this terminology, in [1] the collections of non-equivalent realizable rotation systems on n elements were reported, for all $n \leq 9$.

Thus the set $\mathcal{M}_n$ of non-equivalent realizable rotation systems on n elements can be obtained from $\mathcal{E}_n$: if for some rotation R, both R and R^{-1} are in $\mathcal{E}_n$, we remove one of them. Similarly, $\mathcal{E}_n$ can be easily obtained from $\mathcal{M}_n$. First grow $\mathcal{M}_n$ by adding the inverse of each of its elements, and then run an isomorphism check to get rid of duplicates.

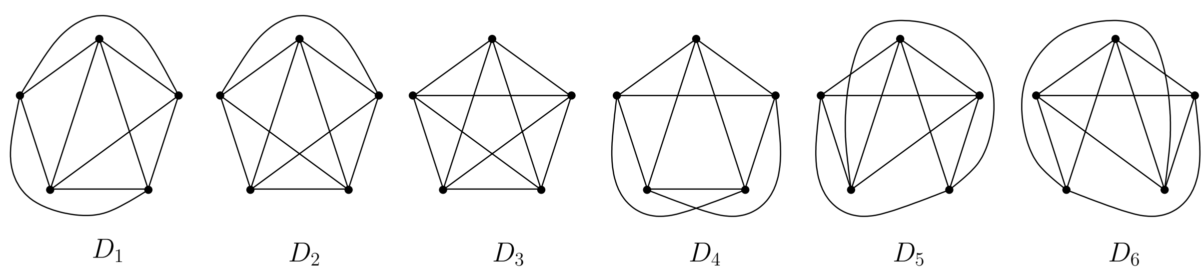

Figure 2: The six non-isomorphic drawings of K_5. Here we adopt the point of view that two drawings are isomorphic if there is an orientation-preserving self-homeomorphism of the sphere that takes one into the other. If we dropped the orientation-preserving condition, then D_5 and D_6 would be isomorphic.

We wrote code to obtain $\mathcal{E}_7$, proceeding as follows. First we obtain $\mathcal{E}_5$. To achieve this, it suffices to take the collection of non-isomorphic drawings of K_5. Here we use the notion that two drawings are *isomorphic* if there is an orientation-preserving self-homeomorphism of the plane that takes one into the other. An easy exercise shows that there are exactly six non-isomorphic drawings of K_5, namely the ones depicted in Figure 2. The class $\mathcal{E}_5$ consists of the rotation systems that correspond to these drawings.

Aichholzer (private communication) noted, based on his results, that a rotation system on 6 elements is realizable if and only if each of its rotation subsystems on 5 elements is realizable. As Kynčl observed in [29, Sect. 1], it follows from this observation and [29, Theorem 1] that a rotation system on $n \geq 5$ elements is realizable if and only if each of its rotation subsystems on 5 elements is realizable.

From this last important observation it follows that the task of finding $\mathcal{E}_6$ is straightforward. For each rotation in $\mathcal{E}_5$, we try all possible ways to extend it to a rotation system on 6 elements, and for each of these possible ways, we test whether or not each of its rotation subsystems on 5 elements is in $\mathcal{E}_5$. Finally, we do an isomorphism check to get rid of duplicates, and finally obtain $\mathcal{E}_6$. To obtain $\mathcal{E}_7$ from $\mathcal{E}_6$ we follow an analogous procedure.

The family $\mathcal{E}_6$ has 165 elements, and $\mathcal{E}_7$ has 22,730 elements. From these lists we generated $\mathcal{M}_6$ and $\mathcal{M}_7$, which have 102 and 11,556 elements, respectively. These coincide with the collections reported in [1, Table 1], as kindly verified by Aichholzer (private communication). The sets $\mathcal{E}_6$ and $\mathcal{E}_7$ are available at `https://orion.math.iastate.edu/lidicky/pub/hill/`.

4.2 Convex rotation systems on 8 elements

Arroyo, McQuillan, Richter, and Salazar [7] have characterized convex drawings of K_n as follows. A good drawing D of K_n, with $n \geq 5$, is convex if and only if all its induced drawings of K_5 are isomorphic to rectilinear drawings. It is well-known that up to isomorphism there are three such drawings of K_5, namely D_1, D_2, and D_3 in Figure 2.

Thus in order to generate the collection $\mathcal{C}_n$ of convex rotation systems, for $n \geq 5$, it suffices to follow the procedure described above to obtain $\mathcal{E}_n$, but in this case the foundation $\mathcal{C}_5$ consists of the rotation systems that correspond to D_1, D_2, and D_3. In this way we constructed $\mathcal{C}_6, \mathcal{C}_7$, and $\mathcal{C}_8$. This last collection consists of 7,360 rotation systems, thus being even more manageable, for a flag algebras treatment, than $\mathcal{E}_7$.

We note that we do not really need the full characterization from [7]. We only need the easy "only if" part, which is readily verified by checking that D_4, D_5, and D_6 are not convex. If we did not have the "if" part, we would still know that the class $\mathcal{C}_8$ we constructed contains the class of convex drawings. Thus our results, in particular Theorem 2, would still hold without this non-trivial direction of the characterization from [7].

5 Flag algebras

This section contains a brief introduction to the flag algebras framework, in the setting of rotation systems. For a more detailed and general exposition, see the original paper of Razborov [38]. For more accessible introductions to flag algebras, see for instance [10, 40].

Throughout this discussion $\mathcal{R}$ is an infinite set of rotation systems, and for each $\ell \in \mathbb{N}$, $\mathcal{R}_\ell$ is the set of all rotations in $\mathcal{R}$ with ℓ elements. For our cases of interest, in the next section we will take $\mathcal{R}$ to be the collection $\mathcal{E}$ of all realizable rotation systems (to prove Theorem 1), or the collection $\mathcal{C}$ of all convex rotation systems (to prove Theorem 2).

For $R \in \mathcal{R}_\ell$ and $R' \in \mathcal{R}_{\ell'}$, define $p(R, R')$ to be the probability that choosing ℓ vertices uniformly at random from R' induces a rotation isomorphic to R. Note that $p(R, R') = 0$ if $\ell > \ell'$.

For $R \in \mathcal{R}$, we denote by $V(R)$ the ground set of R. We use $V(R)$ to hint that we think of the ground elements of R as, and call them, *vertices* (after all, we are interested in rotation systems that are induced by drawings of K_n). Although evidently R is not a graph, the rotation systems that we will investigate come from drawings of K_n, and as such, have an identity as vertices. We let $v(R) := |V(R)|$. Note that $v(R)$ is also the number of elements (cyclic permutations) of R.

We start by defining algebras $\mathcal{A}$ and $\mathcal{A}^\sigma$, where σ is any rotation system in $\mathcal{R}$. These algebras will be called *flag algebras*. Let $\mathbb{R}\mathcal{R}$ be the set of all formal linear combinations of elements in $\mathcal{R}$ with real coefficients. Furthermore, let $\mathcal{K}$ be the linear subspace generated by all linear combinations of the form

$$R - \sum_{R' \in \mathcal{R}_{v(R)+1}} p(R, R') \cdot R'. \tag{3}$$

We define $\mathcal{A}$ as the space $\mathbb{R}\mathcal{R}$ factorized by $\mathcal{K}$. The space $\mathcal{A}$ comes with naturally defined operations of addition, and multiplication by a real number. To introduce the multiplication in $\mathcal{A}$, we first define multiplication of two elements in $\mathcal{R}$. For $R_1, R_2 \in \mathcal{R}$, and $R \in \mathcal{R}_{v(R_1)+v(R_2)}$, we define $p(R_1, R_2; R)$ to be the probability that for a randomly chosen subset I_1 of $V(R)$ of size $v(R_1)$, the rotation subsystems of R induced by I_1 and $I_2 := V(R) \setminus I_1$ are isomorphic to R_1 and R_2, respectively. We set

$$R_1 \times R_2 = \sum_{R \in \mathcal{R}_{v(R_1)+v(R_2)}} p(R_1, R_2; R) \cdot R.$$

The multiplication in $\mathcal{R}$ has a unique linear extension to $\mathbb{R}\mathcal{R}$, which yields a well-defined multiplication also in $\mathcal{A}$. A formal proof of this can be found in [38, Lemma 2.4].

Now we introduce an algebra $\mathcal{A}^\sigma$ for each $\sigma \in \mathcal{R}$. The element σ is usually called a *type* within the flag algebras framework. Without loss of generality, assume that the vertices of σ are labelled $1, 2, \ldots, v(\sigma)$. Define $\mathcal{R}^\sigma$ to be the set of all elements in $\mathcal{R}$ with a fixed *embedding* of σ, i.e., an injective mapping θ from $V(\sigma)$ to $V(R)$ such that the image of θ, denoted by $\theta(V(\sigma))$, induces in R a rotation isomorphic to σ. Following the customary flag algebras terminology, the elements of $\mathcal{R}^\sigma$ are *σ-flags*, and the rotation induced by $\theta(V(\sigma))$ is the *root* of a σ-flag.

For every $\ell \in \mathbb{N}$, we define $\mathcal{R}^\sigma_\ell \subset \mathcal{R}^\sigma$ to be the set of the σ-flags from $\mathcal{R}^\sigma$ that have size ℓ. Analogously to the case for $\mathcal{A}$, for two σ-flags $R, R' \in \mathcal{R}^\sigma$ with embeddings of σ given by θ, θ', we set $p(R, R')$ to be the probability that a randomly chosen subset of $v(R) - v(\sigma)$ ground elements in $V(R') \setminus \theta'(V(\sigma))$ together with $\theta'(V(\sigma))$ induces a substructure that is isomorphic to R through an isomorphism f that preserves the embedding of σ. In other words, the isomorphism f has to satisfy $f(\theta') = \theta$. Let $\mathbb{R}\mathcal{R}^\sigma$ be the set of all formal linear combinations of elements of $\mathcal{R}^\sigma$ with real coefficients, and let $\mathcal{K}^\sigma$ be the linear subspace of $\mathbb{R}\mathcal{R}^\sigma$ generated by all the linear combinations of the form

$$R - \sum_{R' \in \mathcal{R}^\sigma_{v(R)+1}} p(R, R') \cdot R'.$$

We define $\mathcal{A}^\sigma$ to be $\mathbb{R}\mathcal{R}^\sigma$ factorized by $\mathcal{K}^\sigma$.

We now proceed to define the multiplication of two elements from $\mathcal{R}^\sigma$. Let $R_1, R_2 \in \mathcal{R}^\sigma$, $R \in \mathcal{R}^\sigma_{v(R_1)+v(R_2)-v(\sigma)}$, and let θ be the fixed embedding of σ in R. Choose uniformly at random a subset of X in $V(R)\setminus\theta(V(\sigma))$ of size $v(R_1)-v(\sigma)$. Let $Y=V(R)\setminus\{\theta(V(\sigma))\cup Y\}$ of size $v(R_2)-v(\sigma)$. We define $p(R_1,R_2;R)$ to be the probability that $X\cup\theta(V(\sigma))$ and $Y\cup\theta(V(\sigma))$ induce rotations isomorphic to R_1 and R_2, respectively. This definition naturally extends to $\mathcal{A}^\sigma$.

Consider an infinite sequence $(R_n)_{n\in\mathbb{N}}$, where $R_n\in\mathcal{R}_n$. We note that the density $d(R;R_n)$ used in Section 3 is simply $p(R,R_n)$ in the current setting. We use $p(R,R_n)$ in this section as this is the custom notation in flag algebras discussions. We recall from Section 3 that $(R_n)_{n\in\mathbb{N}}$ is *convergent* if the sequence $\big(p(R,R_n)\big)_{n\in\mathbb{N}}$ converges for every $R\in\mathcal{R}$. A standard compactness argument using Tychonoff's theorem yields that every infinite sequence has a convergent subsequence. Fix a convergent sequence $(R_n)_{n\in\mathbb{N}}$. For every $R\in\mathcal{R}$, we set $\phi(R)=\lim_{n\to\infty}p(R,R_n)$ and linearly extend ϕ to $\mathcal{A}$. We usually refer to the mapping ϕ as the *limit of the sequence*. The obtained mapping ϕ is a homomorphism from $\mathcal{A}$ to $\mathbb{R}$. Note that for every $R\in\mathcal{R}$ we have $\phi(R)\geq 0$. Let $\mathrm{Hom}^+(\mathcal{A},\mathbb{R})$ be the set of all such homomorphisms, i.e., the set of all homomorphisms ψ from the algebra $\mathcal{A}$ to $\mathbb{R}$ such that $\psi(R)\geq 0$ for every $R\in\mathcal{R}$. An interesting, crucial fact in the theory of flag algebras, is that this set is exactly the set of all limits of convergent sequences in $\mathcal{R}$ [38, Theorem 3.3].

It is possible to define a homomorphism ϕ^σ from $\mathcal{A}^\sigma$ to $\mathbb{R}$ and an *unlabelling operator* $[\![\cdot]\!]_\sigma:\mathcal{A}^\sigma\to\mathcal{A}$ such that if $\phi^\sigma(A^\sigma)\geq 0$ for some $A^\sigma\in\mathcal{A}^\sigma$, then $\phi([\![A^\sigma]\!]_\sigma)\geq 0$. For details, see [38]. The unlabelling operator is very useful for generating non-obvious valid inequalities of the form $\phi(A)\geq 0$ for some $A\in\mathcal{A}$. In particular, $\phi([\![(A^\sigma)^2]\!]_\sigma)\geq 0$ is always a valid inequality, and the generation of these inequalities can be somewhat automated.

6 Proof of Theorem 3

Proof of Theorem 3. We use the flag algebras framework developed in the previous section, performing the calculations on $\mathcal{E}_7$. As we observed in Section 4, this set has cardinality 22,730. We follow the convention from the previous section to think of the elements in the ground set of a rotation as *vertices*.

We used 1803 labeled flags of 8 types $\sigma_1,\dots,\sigma_8$. Type σ_1 is one labeled vertex and let F_1 be $\mathcal{E}_4^{\sigma_1}$. Type σ_2 are three labeled vertices and let F_2 be $\mathcal{E}_5^{\sigma_2}$. Types σ_i for $3\leq i\leq 8$ are all labeled rotations on 5 vertices, namely the ones associated to the drawings in Figure 2. For $3\leq i\leq 8$, let $F_i=\mathcal{E}_6^{\sigma_i}$. Notice that for all i we picked the sizes of flags in F_i such that the product of any two flags from F_i can be expressed in $\mathcal{E}_7^{\sigma_i}$, and hence subsequently gives an equation in $\mathcal{E}_7$.

The following holds for any $\phi\in\mathrm{Hom}^+(\mathcal{A},\mathbb{R})$. Let $M_1,\dots,M_8$ be positive semidefinite matrices, where M_i has the same dimension as F_i for all i. Then

$$0\leq\phi\left(\sum_{1\leq i\leq 8}[\![F_i^TM_iF_i]\!]_{\sigma_i}\right)=\phi\left(\sum_{R\in\mathcal{E}_7}c_R\cdot R\right),\tag{4}$$

where c_R is a real number depending on $M_1,\dots,M_8$ for each R. The expression (3) implies that

$$\phi\left(N_4\right) = \phi\left(\sum_{R\in\mathcal{E}_7} p(N_4,R)\cdot R\right).$$

By combining this and (4) we obtain the following, where (we recall from Section 2) N_4 is the rotation system that corresponds to :

$$\phi\left(N_4\right) = \phi\left(\sum_{R\in\mathcal{E}_7} p(N_4,R)\cdot R\right) \le \phi\left(\sum_{R\in\mathcal{E}_7} (p(N_4,R)+c_R)\cdot R\right).$$

Let A be as in the statement of Theorem 3. By solving an instance of a semidefinite program, we found $M_1,\ldots,M_8$ such that

$$p(N_4,R)+c_R \le A$$

for all $R\in\mathcal{E}_7$. Noting that $\phi\left(\sum_{R\in\mathcal{E}_7} R\right) = 1$, we obtain

$$\phi\left(N_4\right) \le \phi\left(\sum_{R\in\mathcal{E}_7} (p(N_4,R)+c_R)\cdot R\right) \le A\cdot\phi\left(\sum_{R\in\mathcal{E}_7} R\right) = A.$$

Let $R_1, R_2, \ldots$ be a convergent sequence of realizable rotation systems. Since $\phi(N_4) = \lim_{i\to\infty} p(N_4,R_i) = \lim_{i\to\infty} d(N_4;R_i)$, this last equation implies that $\lim_{i\to\infty} d(N_4;R_i) \le A < 0.630400393$, as claimed in Theorem 3.

Due to space limitations, we provide $\mathcal{E}_7$, F_i and M_i for all i, as well as programs that perform the calculations, in electronic files at `https://orion.math.iastate.edu/lidicky/pub/hill/`. □

Proof of Theorem 2. In this case we performed the calculations on $\mathcal{C}_8$. We used 3664 labeled flags of 5 types $\sigma_1,\ldots,\sigma_5$. Type σ_1 is one labeled vertex and let F_1 be $\mathcal{C}_4^{\sigma_1}$, i.e., all realizable convex rotation systems on 4 vertices, where one vertex is labeled. Type σ_2 are three labeled vertices and let F_2 be $\mathcal{C}_5^{\sigma_2}$. Types σ_i for $3\le i\le 5$ are all labeled rotations on 5 vertices, namely the ones associated to the drawings D_1, D_2, and D_3 in Figure 2. For $3\le i\le 5$, let $F_i = \mathcal{C}_6^{\sigma_i}$. Notice that for all i we picked the sizes of flags in F_i such that the product of any two flags from F_i can be expressed in $\mathcal{C}_8^{\sigma_i}$, and hence subsequently gives an equation in $\mathcal{C}_8$.

We can now pick up the proof of Theorem 1 at the paragraph that starts "The following holds…", with the following changes. Instead of having positive semidefinite matrices $M_1,\ldots,M_8$, we have only five positive semidefinite matrices $M_1,\ldots,M_5$ (here again each M_i has the same dimension as F_i). The first summation in (4) is now on $1\le i\le 5$, and every summation on $R\in\mathcal{E}_7$ gets replaced by a summation on $R\in\mathcal{C}_8$. Finally, instead of the constant A in Theorem 1, we have the constant B in Theorem 2.

With these changes the proof carries over exactly as in the previous proof, finally obtaining that $\lim_{i\to\infty} d(N_4;R_i) \le B < 0.627285406$.

Due to space limitations, we provide $\mathcal{C}_8$, F_i and M_i for all i, as well as programs that perform the calculations, in electronic files at `https://orion.math.iastate.edu/lidicky/pub/hill/`. □

7 Concluding remarks

As we mentioned in Section 1, the flag algebras framework was used by Norin and Zwols [34] to attack another crossing number problem, namely Zarankiewicz's conjecture. Recently, Goaoc, Hubard, De Joannis De Verclos, Sereni, and Volec [21] also used flag algebras to approach a related problem in discrete geometry, namely the density of k-tuples in convex position in point sets in the plane.

Norin and Zwols computed all the good drawings of $K_{3,4}$, and for each such drawing they recorded which pairs of edges cross each other. With this information, they used flag algebras to obtain the lower bound $\lim_{n\to\infty} \mathrm{cr}(K_{n,n})/Z(n,n) > 0.905$. In this paper we worked with rotation systems, but we note that this approach is equivalent to the alternative (à la Norin-Zwols) of computing all good drawings of K_7 and recording, for each such drawing, which pairs of edges cross each other. This follows since from the rotation system of a drawing one can tell which pairs of edges cross each other in the drawing [9, 20].

An earlier approach we tried involved associating to a good drawing $\mathcal{D}$ of K_m the 4-uniform hypergraph $\mathcal{H}_{\mathcal{D}}$ whose vertices are the vertices of the drawing, and where four vertices form an edge if and only if the drawing of K_4 induced from $\mathcal{D}$ on these four vertices has a crossing. We refer the reader to [42, Section 13.4] for a discussion on the connection between crossing number problems and Turán-type hypergraph problems. This approach, also using flag algebras, yielded a considerably weaker lower bound than the one in Theorem 1. Obtaining poorer bounds in this setting is quite natural since, as we recalled above, with the rotation system of a drawing one can tell not only which K_4s have a crossing, but exactly which edges cross each other in a given K_4.

We are currently working on two separate approaches to apply flag algebras to obtain improved lower bounds on the rectilinear crossing number $\overline{\mathrm{cr}}(K_{n,n})$. We can currently show that $\lim_{n\to\infty} \overline{\mathrm{cr}}(K_{n,n})/Z(n,n) > 0.973$, and we hope to get an even better lower bound when a set of ongoing calculations is completed. Together with Pfender and Norin, we had previously considered the special version of rectilinear drawings in which the partite classes are separated by a line. In this case, we got a lower bound of 0.99.

Let us mention that it might be possible to improve the constants A and B in Theorems 3 and 4 by a tiny amount. The matrices M_i in the proofs of these theorems were first obtained by a semidefinite programming solver. These matrices do not contain exact entries, and some small rounding was necessary to ensure that the M_is are indeed positive semidefinite and the evaluation of $p(N_4, R) + c_R$ does not have any numerical errors. We have not tried to optimize the rounding process as we think the possible improvement is negligible.

For Theorem 3, performing the calculations on $\mathcal{E}_8$ would likely provide a remarkable improvement. Unfortunately, the size of this set makes it out of reach for current computers. Similarly, for Theorem 4, performing the calculations on $\mathcal{C}_9$ would very likely result in a considerable improvement, but this set is also too big to be handled with computer power available at this time.

Aichholzer (private communication) has verified that all crossing-minimal drawings of K_n, for $n \leq 12$, are convex. Thus it seems reasonable to conjecture that all crossing-minimal drawings of K_n, for every integer n, are convex. If this were proved, the bound in Theorem 2 would apply for the crossing number of K_n.

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