On a conjecture concerning 4-coloring of graphs with one crossing

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Abstract

We conjecture that every graph of minimum degree five with no separating triangles and drawn in the plane with one crossing is 4-colorable. In this paper, we use computer enumeration to show that this conjecture holds for all graphs with at most 28 vertices, explore the consequences of this conjecture and provide some insights on how it could be proved.

Famously, every planar graph is 4-colorable [1, 2]. It is natural to ask whether this statement can be strengthened. As shown in [7, 9], for every surface Σ other than the sphere, there are infinitely many minimal non-4-colorable, i.e., 5-*critical*, graphs which can be drawn in Σ (a graph G is k-*critical* if $\chi(G) = k$ and every proper subgraph of G is $(k-1)$-colorable). Still, one could hope that at least for simplest non-trivial surfaces Σ (projective plane, torus), it might be possible to characterize 5-critical graphs drawn in Σ, or at least to develop a polynomial-time algorithm to test 4-colorability of graphs drawn in Σ. However, either of these goals seems far beyond our reach using the current methods.

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To obtain at least some intuition, we consider "minimally nonplanar" graphs, or more precisely, graphs that can be drawn in the plane with at most one crossing; let $\mathcal{C}$ denote the class of all such graphs. Let us remark that graphs in $\mathcal{C}$ can be drawn without crossings both in the projective plane and on the torus. It is still easy to construct infinite families of 5-critical graphs in $\mathcal{C}$; however, preliminary computer-assisted investigations suggest that obtaining their characterization could be possible. In this paper, we explore in detail a natural conjecture arising from these investigations.

Of course, 5-critical graphs have minimum degree at least four. Moreover, there is a natural way how to handle vertices of degree four: Suppose $G \neq K_5$ is a 5-critical graph and a vertex $v \in V(G)$ has degree four. Since G is 5-critical and $G \neq K_5$, the graph G does not contain K_5 as a subgraph, and thus v has distinct neighbors x and y such that $xy \notin E(G)$. Let G' be the graph obtained from $G - v$ by identifying x with y to a single vertex. Assuming that G can be drawn in the plane with one crossing so that this crossing is not on the edges vx and vy, the graph G' can also be drawn in the plane with at most one crossing. Moreover, it is easy to see that any 4-coloring of G' could be extended to a 4-coloring of G. Since G is 5-critical, it follows that G' is not 4-colorable, and thus it has a 5-critical subgraph $G'' \in \mathcal{C}$. Based on this observation, one could hope to inductively prove statements about the structure of 5-critical graphs in $\mathcal{C}$ of minimum degree four. Indeed, a similar inductive argument used to handle 4-faces has been crucial in the development of the theory of 4-critical triangle-free graphs, which eventually lead to design of a linear-time algorithm for 3-colorability of triangle-free graphs on surfaces [5].

For this argument to take off, one needs to handle the basic case of 5-critical graphs in $\mathcal{C}$ of minimum degree at least five. Based on computer-assisted experiments, we believe no such graphs exists, and thus K_5 is the sole basic case for the argument described above.

Conjecture 1. *Every 5-critical graph in $\mathcal{C}$ contains a vertex of degree four.*

Indeed, the following stronger statement seems to hold (a subgraph K of a graph G is *separating* if $G - V(K)$ is not connected).

Conjecture 2. *If a graph G does not contain separating triangles and has a drawing in the plane with one crossing such that all vertices not incident with the crossed edges have degree at least five, then G is 4-colorable.*

Let us remark that throughout the paper, the graphs are simple, without loops and parallel edges. It is easy to see that 5-critical graphs do not

contain separating cliques, and thus Conjecture 2 implies Conjecture 1. The assumption that G does not contain separating triangles cannot be dropped: Every graph $G \in \mathcal{C}$ has a supergraph of minimum degree at least five in $\mathcal{C}$, obtained by gluing a copy of the icosahedron at each vertex of G. A similar argument shows that Conjecture 2 has the following consequence.

Lemma 3. *Let $G \in \mathcal{C}$ be a graph without separating triangles, and consider any drawing of G in the plane with at most one crossing. If G is not 4-colorable and Conjecture 2 holds, then there exists a vertex $v \in V(G)$ of degree four such that all incident faces have length three and their boundaries do not contain the crossing.*

In particular, this has the following consequence (where W_k is the *k-wheel* graph consisting of a k-cycle and a vertex adjacent to all vertices of this cycle).

Corollary 4. *If Conjecture 2 holds, then every graph in $\mathcal{C}$ that does not contain W_4 as a subgraph is 4-colorable.*

It is well known that it suffices to prove the Four Color Theorem for triangulations of the plane. Similarly, Conjecture 2 has the following equivalent form. Let $\mathcal{C}_0 \subseteq \mathcal{C}$ consist of graphs obtained from 4-connected plane triangulations by choosing distinct facial triangles xuv and yuv sharing an edge uv and adding the edge xy; we say that the vertices x, y, u, and v are *external* and all other vertices *internal*.

Conjecture 5. *Every non-4-colorable graph in $\mathcal{C}_0$ has an internal vertex of degree four.*

Assuming Conjecture 2 holds, we can show that every non-4-colorable graph in $\mathcal{C}_0$ can be obtained from K_5 by a sequence of simple operations. Since we want to avoid creating separating triangles, this is not completely straightforward; see Section 3 for the definition of the *expansion* operation.

Theorem 6. *If Conjecture 2 holds, then for every non-4-colorable graph $G \in \mathcal{C}_0$, there exists a sequence $K_5 = G_0, G_1, \ldots, G_m = G$ of graphs such that for every $i \in \{1, \ldots, m\}$, the graph $G_i \in \mathcal{C}_0$ is non-4-colorable and is obtained from G_{i-1} by expansion.*

The expansion operation increases the number of vertices, and thus Theorem 6 can be turned into a relatively efficient algorithm to enumerate the non-4-colorable graphs in $\mathcal{C}_0$. However, it needs to be noted that there are 5-critical graphs in $\mathcal{C}$ that do not belong to $\mathcal{C}_0$; see Figure 1 for an example.

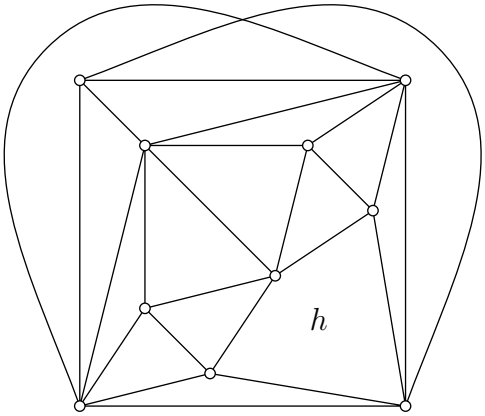


Figure 1: A 5-critical graph in $\mathcal{C}$ with a 4-face h.

Consequently, Theorem 6 is somewhat less explicit than it might appear, since one of the options for the expansion operation needs to be essentially "if G_{i-1} contains the subgraph depicted in Figure 1, then G_i can be obtained by replacing the part of G_{i-1} drawn inside the face h by any larger triangulation without separating triangles".

The rest of the paper is organized as follows. In Section 1, we give some basic facts on 5-critical graphs in $\mathcal{C}$, relating them to plane graphs which are not 4-colorable so that the coloring of a specific 4-face matches a prescribed pattern, and prove Lemma 3 and the equivalence of Conjectures 2 and 5. In Section 2, we give results on enumeration of planar graphs without separating triangles and with all vertices of degree at most four incident with a specific face. Using these results, we give a computer-assisted proof that Conjecture 2 holds for graphs with at most 28 vertices. In Section 3, we prove Theorem 6. Finally, in Section 4, we investigate how the proof of the Four Color Theorem could be modified to prove Conjecture 2.

1 Reduction to precoloring extension

Let us start with a simple observation.

Lemma 7. *If a graph $G \in \mathcal{C}$ is not 4-colorable, then G is non-planar and for any drawing of G in the plane with one crossing, the vertices of G incident with the crossing edges induce a clique.*

Proof. The graph G is clearly non-planar by the Four Color Theorem. Con-

sider any drawing of G in the plane with one crossing, let u_1v_1 and u_2v_2 be the crossed edges in this drawing, and suppose for a contradiction that $\{u_1, u_2, v_1, v_2\}$ is not a clique in G. By symmetry, we can assume that the vertices u_1 and u_2 are non-adjacent in G. Let G' be the graph obtained from G by identifying u_1 with u_2 to a single vertex u. Observe that G' is planar, and thus it has a proper 4-coloring. The 4-coloring of G' corresponds to a 4-coloring of G obtained by giving u_1 and u_2 the color of u. This is a contradiction, since G is non-4-colorable. □

Consider a graph G drawn in the plane with one crossings, let u_1v_1 and u_2v_2 be the crossed edges in this drawing, and suppose $\{u_1, u_2, v_1, v_2\}$ is a clique in G. We can redraw the edges u_1u_2, u_2v_1, v_1v_2, and v_2u_1 next to the edges u_1v_1 and u_2v_2 without introducing any crossings, so that the 4-cycle $C = u_1u_2v_1v_2$ bounds a face of the corresponding plane drawing of the graph $G_0 = G - \{u_1v_1, u_2v_2\}$. Clearly any 4-coloring of G assigns pairwise different colors to the vertices of C, and thus G is 4-colorable if and only if there exists a 4-coloring of G_0 using all four colors on C.

The *type* of a coloring ψ of a graph F is the equivalence $\approx$ such that for $u, v \in V(F)$, we have $u \approx v$ if and only if $\psi(u) = \psi(v)$. Note that there are exactly four types of 4-colorings of a 4-cycle $K = x_1x_2x_3x_4$; such a 4-coloring φ is

- *rainbow* if $\varphi(x_1) \neq \varphi(x_3)$ and $\varphi(x_2) \neq \varphi(x_4)$,
- x_i*-diagonal* for $i \in \{1, 2\}$ if $\varphi(x_i) = \varphi(x_{i+2})$ and $\varphi(x_{3-i}) \neq \varphi(x_{5-i})$, and
- *bichromatic* if $\varphi(x_1) = \varphi(x_3)$ and $\varphi(x_2) = \varphi(x_4)$.

Let H be a plane graph and let F be the subgraph of H consisting of the vertices and edges incident with the outer face of H. For a type $\approx$ of colorings of F, we say that a 4-coloring ψ of H *matches* $\approx$ if $\approx$ is the type of the restriction of ψ to F. We say that a graph H is $\approx$*-forbidding* if H does not have any 4-coloring matching $\approx$. Equivalently, for any 4-coloring φ of F, if the type of φ is $\approx$, then φ does not extend to a 4-coloring of H.

A plane graph G with the outer face bounded by a 4-cycle K is *rainbow-forbidding* if G is $\approx$-forbidding for the type $\approx$ of rainbow colorings of K. Similarly, we define the notion of an x*-diagonal-forbidding* graph for $x \in V(K)$ and of a *bichromatic-forbidding* graph. With these definitions in place, we can restate Lemma 7 as follows.

Corollary 8. *A graph G is not 4-colorable and belongs to $\mathcal{C}$ if and only if there exists a rainbow-forbidding plane graph H with the outer face bounded by a 4-cycle $u_1u_2v_1v_2$ such that G is isomorphic to the graph obtained from H by adding the edges u_1v_1 and u_2v_2.*

Thus, the problem of characterizing non-4-colorable graphs in $\mathcal{C}$ is equivalent to the rainbow precoloring extension problem in plane graphs with the outer face of size four. Next, let us note a standard observation on triangulating planar graphs.

Lemma 9. *Let H be a plane graph such that every triangle bounds a face and the outer face has length greater than three. Then there exists a supergraph H' of H such that $V(H') = V(H)$, the outer face of H is also the outer face of H', all internal faces of H' are triangles, and every triangle in H' bounds a face.*

Proof. Let us argue that if f is an internal face of H of length at least four, then there exist distinct non-adjacent vertices u and v incident with f such that every triangle in $H + uv$ (with the edge uv drawn inside f) bounds a face.

- If the boundary of f is disconnected, then we can simply add an edge between different components; such an edge clearly is not contained in any triangle in the resulting graph, and thus every triangle still bounds a face.

- Otherwise, let $W = v_1v_2 \ldots v_k$ be the closed walk bounding f. If say v_2 is a cut vertex separating v_1 from v_3 in H, then we can let $u = v_1$ and $v = v_3$; the only triangle in $H + v_1v_3$ containing the edge v_1v_3 is $v_1v_2v_3$, and the edge v_1v_3 can be drawn inside f so that this triangle bounds a face.

- Hence, we can assume that W does not contain such a cut vertex, and thus W is a cycle. Suppose that say v_2v_i, where $i \in \{4, 5, \ldots, k\}$, is an edge of H. By planarity, v_1 and v_3 are not adjacent in H. Moreover, the vertices v_1 and v_3 have no common neighbors except for v_2 if $k \geq 5$ and except for v_2 and v_4 if $k = 4$. Hence, every triangle in $H + v_1v_3$ bounds a face.

- Consequently, we can assume that W is an induced cycle. Suppose next that each pair of distinct non-consecutive vertices of W has a common neighbor not in W. By planarity, it follows that all vertices

in W have a common neighbor outside of W. However, since the outer face of H has length greater than three, this implies that H contains a non-facial triangle, which is a contradiction.

- Therefore, there exist distinct non-consecutive vertices u and v of W that do not have a common neighbor outside of W. Then all triangles in $H + uv$ bound faces.

Thus, we can repeatedly add edges to H while preserving its outer face and the invariant that every triangle bounds a face, until all internal faces become triangles. □

Using this lemma, we can reformulate Conjecture 2 in terms of precoloring extension in plane graphs (a vertex of a plane graph is *internal* if it is not incident with the outer face, and *external* otherwise); as well as prove the equivalence with Conjecture 5.

Theorem 10. *The following claims are equivalent for every positive integer n:*

(i) *Let G be a graph with $|V(G)| = n$ and without separating triangles. If G has a drawing in the plane with one crossing such that all vertices not incident with the crossed edges have degree at least five, then G is 4-colorable.*

(ii) *Let H' be a plane graph with $|V(H')| = n$ such that every triangle in H' bounds a face and all internal faces of H' are triangles. If H' is rainbow-forbidding, then H' has an internal vertex of degree at most four.*

(iii) *Let H be a plane graph with $|V(H)| = n$ such that every triangle in H bounds a face. If H is rainbow-forbidding, then H has an internal vertex of degree at most four.*

(iv) *Every non-4-colorable graph $G \in \mathcal{C}_0$ with n vertices has an internal vertex of degree four.*

Proof. We prove several implications:

(i)⇒(ii) Let H' be a rainbow-forbidding plane graph with n vertices such that every triangle in H' bounds a face and all internal faces of H' are triangles. Let $K = x_1x_2x_3x_4$ be the 4-cycle bounding the outer face of H'. Note that the cycle K is induced: Otherwise, since every

triangle in H' bounds a face, H' would consist of K together with one additional edge, and H' would not be rainbow-forbidding. Consider the graph $G = H' + \{x_1x_3, x_2x_4\}$, drawn in the plane so that only the edges x_1x_3 and x_2x_4 cross.

We claim that G does not contain any separating triangles. Indeed, consider any triangle T in G. By symmetry, we can assume that T does not contain the edge x_2x_4, and thus $H' \cup T$ is a plane triangulation. In particular, $H' \cup T$ is 3-connected. If T is a separating triangle in $H' \cup T$, then the planarity and 3-connectedness implies that $H' - V(T)$ has exactly two components K_1 and K_2, each induced by all vertices of H' drawn in one of the faces of T. Since every triangle in H' bounds a face, it follows that $x_1x_3 \in E(T)$, $x_2 \in V(K_1)$ and $x_4 \in V(K_2)$. However, since the edge x_2x_4 of G connects K_1 with K_2, this implies that $G - V(T)$ has only one component, and thus the triangle T is not separating.

Since H' is rainbow-forbidding, the graph G is not 4-colorable, and thus by (i), there exists a vertex $v \in V(G) \setminus V(K)$ such that $4 \geq \deg_G v = \deg_{H'} v$. Hence, (i) implies (ii).

(ii)⇒(iii) Consider a rainbow-forbidding plane graph H with n vertices such that every triangle bounds a face, and let $K = x_1x_2x_3x_4$ be the 4-cycle bounding the outer face of H. By Lemma 9, there exists a supergraph H' of H such that $V(H') = V(H)$, the outer face of H' is bounded by K, all internal faces of H' are triangles, and every triangle in H' bounds a face. Clearly H' is also rainbow-forbidding. By (ii), H' has an internal vertex v of degree at most four. Clearly v is also an internal vertex of H and its degree in H is at most as large as in H'. Hence, (ii) implies (iii).

(iii)⇒(i) Suppose now that G is a graph such that $|V(G)| = n$, G does not contain separating triangles, and G has a drawing in the plane with one crossing such that all vertices not incident with the crossed edges have degree at least five. If the set X of vertices incident with the crossed edges e_1 and e_2 of G does not induce a clique in G, then G is 4-colorable by Lemma 7.

Otherwise, the graph $H = G - \{e_1, e_2\}$ has a plane drawing where the 4-cycle on X bounds the outer face. All internal vertices of H have the same degree as in G, i.e., at least five. Moreover, every triangle in H bounds a face, as otherwise it would be a separating triangle in G. By (iii), H is not rainbow-forbidding, and thus H has a 4-coloring

φ whose restriction to $H[X]$ has the rainbow type. Then φ is also a proper 4-coloring of G. Therefore, (iii) implies (i).

(iv)⇒(ii) Consider a rainbow-forbidding plane graph H' with n vertices such that every triangle in H' bounds a face and all internal faces of H' are triangles. Let $K = x_1x_2x_3x_4$ be the 4-cycle bounding the outer face of H'. Note that K is an induced cycle, as otherwise we would have $V(H') = V(K)$ and H' would not be rainbow-forbidding. For $i \in \{1, 2\}$, let $G_i = H' + x_ix_{i+2}$, so that G_i is a plane triangulation. If G_i is 4-connected, then $G_i + x_{3-i}x_{5-i} \in \mathcal{C}_0$ has an internal vertex of degree at most four by (iv), and so does H'.

Otherwise, for both $i \in \{1, 2\}$, there must exist a separating triangle in G_i, necessarily containing the edge x_ix_{i+2} since all other triangles bound faces by the assumptions on H'. By planarity, this is only possible if H' contains a vertex v adjacent to $x_1, \ldots, x_4$. Since all triangles in H' bound faces, we conclude that $\deg v = 4$. Therefore, (iv) implies (ii).

Finally, it is clear that (i) implies (iv). □

In particular, to verify Conjecture 2 for graphs with n vertices, it suffices it verify the claim from Theorem 10(ii) for graphs with n vertices. For small n, this can be done by a computer-assisted enumeration, as we discuss in the following section.

Next, let us argue that Conjecture 2 is also equivalent to a claim about diagonal-forbidding graphs (we say that a plane graph is *diagonal-forbidding* if it is x-diagonal-forbidding for a vertex x incident with its outer face). For an integer $n \geq 5$, let $D(n)$ be the following claim:

$D(n)$: For every plane graph H such that $5 \leq |V(H)| \leq n$, all internal faces of H are triangles, and every triangle in H bounds a face, if H is diagonal-forbidding, then H contains an internal vertex of degree at most four.

Let $R(n)$ be the analogous claim for rainbow-forbidding graphs.

Theorem 11. *For every integer $n \geq 5$, $R(n+1)$ implies $D(n)$ and $D(n+1)$ implies $R(n)$.*

Proof. Let us prove that $R(n + 1)$ implies $D(n)$. Let H be a plane graph with at least five but at most n vertices such that all internal faces of H are triangles and every triangle in H bounds a face, with the outer face

bounded by a 4-cycle K. Suppose that H is x-diagonal-forbidding for a vertex $x \in V(K)$. We need to argue that if $R(n+1)$ holds, then H contains an internal vertex of degree at most four.

Note that K is an induced cycle in H, since H has at least five vertices and every triangle in H bounds a face. Furthermore, note that this implies that every vertex of K has degree at least three in H. If $|V(H)| = 5$, then H consists of K and an internal vertex of degree four, and the conclusion holds (and also, H actually is not diagonal-forbidding).

Hence, suppose that $|V(H)| \geq 6$. If $\deg x = 3$, then let $H' = H - x$. Otherwise, let H' be the graph obtained from H by adding a vertex x' drawn in the outer face of H and adjacent to x and the two neighbors of x in K. Observe than in both cases, the outer face of H' is bounded by a 4-cycle, H' is rainbow-forbidding, it has at least five but at most $n+1$ vertices, all internal faces of H' are triangles, and every triangle in H' bounds a face. By $R(n+1)$, there exists an internal vertex v of H' of degree at most four. Note that $v \neq x$, since if $x \in V(H')$, then x has degree at least four in H and at least five in H'. Hence, v is also an internal vertex of H of degree at most four.

The implication $D(n+1) \Rightarrow R(n)$ is proved analogously. □

Together with Theorem 10, this gives the following consequence.

Corollary 12. *The following claims are equivalent:*

- *Conjecture 2 holds.*
- *$D(n)$ holds for every $n \geq 5$.*
- *$R(n)$ holds for every $n \geq 5$.*

Interestingly, the analogous claim for bichromatic-forbidding graphs is false, as we discuss at the end of Section 2.

The proof of Lemma 3 follows an idea similar to the proof of Theorem 10. For a future use, we prove a somewhat stronger claim, which implies Lemma 3 by Lemma 7.

Lemma 13. *Let H be a rainbow- or diagonal-forbidding plane graph such that every triangle in H bounds a face and $|V(H)| \geq 5$. If Conjecture 2 holds, then there exists an internal vertex $v \in V(H)$ of degree four such that all incident faces have length three.*

Proof. For any integer $k \geq 4$, let F_k be a plane graph such that the outer face of F_k is bounded by an induced cycle of length k, all triangles in F_k bound

faces, all internal faces of F_k have length three, and all internal vertices of F_k have degree at least five. For example, we can define F_4 to be the graph obtained from the icosahedron by deleting an edge, and for $k \geq 5$, we can let F_k be the k-wheel. The claim of the lemma essentially follows by filling in each internal face f of H of length $|f| \geq 4$ by a copy of $F_{|f|}$ and applying $D(n)$ or $R(n)$ to the resulting n-vertex graph H'. However, we need to be slightly more careful in order to argue that H' satisfies the assumptions of $D(n)$ or $R(n)$.

Suppose for a contradiction that H is a counterexample to Lemma 13 with the smallest number of vertices. Let K be the 4-cycle bounding the outer face of H. Note that the cycle K is induced, since $|V(H)| \geq 5$ and every triangle in H bounds a face.

Let us consider the case that H contains a separating clique Q. There exists a component C of $H - V[Q]$ such that the subgraph $H_1 = H[V(Q) \cup V(C)]$ satisfies $K \subseteq H_1$. Let $H_2 = H - V(C)$. By the Four Color Theorem, H_2 has a 4-coloring φ_2. Since Q is a clique, for any 4-coloring φ_1 of H_1, we can permute the colors in the coloring φ_2 so that it matches φ_1 on Q, and thus $\varphi_1 \cup \varphi_2$ is a 4-coloring of H. Since H is rainbow- or diagonal-forbidding, it follows that H_1 also is rainbow- or diagonal-forbidding. By the minimality of H, there exists an internal vertex $v \in V(H_1)$ of degree four such that all incident faces are bounded by triangles. Since all triangles in H bound faces, these triangles also bound faces in H, and thus v has degree four in H as well. This is a contradiction, since H is a counterexample.

Therefore, the graph H does not contain any separating clique. A similar argument shows that every internal vertex of H has degree at least four. Since H does not contain separating cliques, it is 2-connected and every face of H is bounded by an induced cycle. Let H' be the plane graph obtained from H by, for each internal face f of length $|f| \geq 4$, adding a copy of $F_{|f|}$ and identifying the cycle bounding its outer face with the boundary cycle of f. Since the cycles bounding the internal faces of H are induced, all triangles created by this modification bound faces, and thus every triangle in H' bounds a face. Since H is rainbow- or diagonal-forbidding, H' is rainbow- or diagonal-forbidding as well. Moreover, all internal faces of H' have length three.

Assuming that Conjecture 2 holds, $R(|V(H')|)$ or $D(|V(H')|)$ applies and consequently H' contains an internal vertex v of degree at most four. Since all internal vertices of the graphs F_k have degree at least five, v is also an internal vertex of H. All internal vertices of H have degree at least four, and thus v has degree exactly four and all faces of H incident with v are triangles (otherwise a copy of a graph $F_{|f|}$ would be glued into a face f

incident with v in the construction of H', increasing the degree of v to at least five). Hence, the conclusion of the lemma holds for v. □

Finally, let us point out a useful fact about non-extendable precolorings in a plane graph G with the outer face bounded by a 4-cycle $K = x_1x_2x_3x_4$. If K has a chord, say $x_1x_3 \in E(G)$, then G is both x_1-diagonal-forbidding and bichromatic-forbidding. However, we claim that at most one type of colorings can be forbidden when K is an induced cycle.

To see this, let us define $\approx_r$, $\approx_1$, $\approx_2$, and $\approx_b$ to be the types of rainbow, x_1-diagonal, x_2-diagonal, and bichromatic 4-colorings of K, let Q be an auxiliary graph formed by the 4-cycle with vertices $\approx_r$, $\approx_1$, $\approx_b$, and $\approx_2$ in order, and let us note the following observation based on Kempe chains.

Lemma 14. *Let G be a plane graph with the outer face bounded by a 4-cycle $K = x_1x_2x_3x_4$, let φ be a 4-coloring of G, and let $\approx$ be the type of the restriction of φ to K. There exists a 4-coloring φ' of G such that the type of the restriction of φ' to K is adjacent to $\approx$ in Q.*

Proof. This follows by the inspection of Kempe chains. Let us illustrate the argument for a coloring φ assigning colors $1, 2, 1, 3$ to vertices of K in order, i.e., for the case that $\approx$ is $\approx_1$. Let A be the subgraph of G induced by vertices of colors 1 and 4, and let B be the subgraph induced by the vertices of colors 2 and 3. If A does not contain a path between x_1 and x_3, then we can exchange colors 1 and 4 on the component of A containing x_3, obtaining a 4-coloring of G whose restriction to K has type $\approx_r$. If B does not contain a path between x_2 and x_4, then we can exchange the colors 2 and 3 on the component of B containing x_4, obtaining a 4-coloring of G whose restriction to K has type $\approx_b$. And, by planarity, G cannot at the same time contain a path between x_1 and x_3 in colors 1 and 4, as well as a path between x_2 and x_4 in colors 2 and 3. □

Next, let us make some observations on the existence of 4-colorings that follow from the Four Color Theorem.

Observation 15. *Let G be a plane graph with the outer face bounded by a 4-cycle $K = x_1x_2x_3x_4$. Then either G has a 4-coloring matching $\approx_r$, or two 4-colorings matching $\approx_1$ and $\approx_2$.*

Proof. By the Four Color Theorem, the planar graph G_1 obtained from G by adding the edge x_2x_4 has a 4-coloring φ_1, and the planar graph obtained from G by adding the x_1x_3 has a 4-coloring φ_2. Note that either at least one of φ_1 and φ_2 matches $\approx_r$, or φ_1 matches $\approx_1$ and φ_2 matches $\approx_2$. □

Observation 16. *Let G be a plane graph with the outer face bounded by an induced 4-cycle $K = x_1x_2x_3x_4$. Then either G has a 4-coloring matching $\approx_b$, or two 4-colorings matching $\approx_1$ and $\approx_2$.*

Proof. This follows similarly to Observation 15 by considering the planar graphs obtained from G by identifying x_1 with x_3 and by by identifying x_2 with x_4. Note that these graphs do not contain loops, since K is an induced cycle. □

The desired claim is now just a simple combination of these observations.

Corollary 17. *Let G be a plane graph with the outer face bounded by an induced 4-cycle $K = x_1x_2x_3x_4$. Then G has three 4-colorings whose restrictions to K have different types.*

Proof. Let A be the set of vertices $\approx \in V(Q)$ such that G has a 4-coloring matching $\approx$. By Observations 15 and 16, either $\approx_1, \approx_2 \in A$, or $\approx_b, \approx_r \in A$. In either case Observation 14 implies that A contains at least one more type. □

2 Generating candidates

A *canvas* is a plane graph with the outer face bounded by a cycle of length at least four and all other faces of length three. A *candidate* is a canvas such that every triangle bounds a face and all internal vertices have degree at least five. A candidate is an *ℓ-candidate* if its outer face has length ℓ. By Theorem 10, in order to verify Conjecture 2 for graphs with at most n vertices, it suffices to enumerate the 4-candidates with at most n vertices and check that none of them is rainbow-forbidding.

In this section, we develop a way to enumerate ℓ-candidates for any fixed $\ell \geq 4$. More precisely, we are going to describe a system of reductions which turn each sufficiently large ℓ-candidate into a smaller one. Thus, all ℓ-candidates can be produced from small ℓ-candidates by a sequence of the inverse operations to these reductions.

A cycle K in a canvas is *hollow* if no vertex is drawn in the open disk bounded by K (but edges can be drawn in this disk). A candidate is *internally k-connected* if all cycles of length less than k other than the one bounding the outer face are hollow. In particular, every candidate is internally 4-connected. For a canvas G and a cycle C in G, the subgraph G' of G drawn in the closed disk bounded by C is a *subcanvas* of G. When G is a candidate, we also say that G' is a *subcandidate*, and it is an *m-subcandidate*

if C has length m. When we look for a reducible configuration, it is convenient to assume that the candidate is internally 5-connected, since then its reduction turns out to be guaranteed not to create non-facial triangles. We can do so by focusing on the smallest 4-subcandidate H of the considered ℓ-candidate G; the minimality of H clearly implies that H is internally 5-connected. In order for the reducible configuration found in H to be valid in G, we need to ensure that the degrees of external vertices of H are not decreased below five when the reduction is performed in G. To facilitate this, we are going to require that in the canvas resulting from the reduction in H, all external vertices have degree at least four. Thus, we say that a canvas H is *thick* if its outer face has length at least five or all external vertices of H have degree at least four. Let us argue that this condition is automatically satisfied by non-trivial internally 5-connected candidates.

Observation 18. *Every internally* 5*-connected* ℓ*-candidate* H *with at least one internal vertex is thick.*

Proof. By the definition of thickness, this is trivial if $\ell \geq 5$. Hence, suppose that $\ell = 4$, and let $K = v_1v_2v_3v_4$ be the cycle bounding the outer face of H. Suppose for a contradiction that H is not thick, and thus say $\deg v_1 \leq 3$. Let C be the cycle in H formed by the path on the neighbors of v_1 and the path $v_2v_3v_4$; we have $|V(C)| = \deg v_1 + 1 \leq 4$. Observe that every vertex of H except for v_1 is drawn in the closed disk bounded by C. However, the candidate H is internally 5-connected, and thus the cycle C is hollow. Since H has at least one internal vertex, we conclude that $\deg v_1 = 3$ and H has exactly one internal vertex v. However, then $\deg v \leq |V(K)| = 4$, contradicting the assumption that H is a candidate. □

We are also going to need the following simple discharging argument.

Lemma 19. *Every thick* 4*-candidate contains an internal vertex of degree five with no external neighbor.*

Proof. Let G be a thick 4-candidate and let K be the 4-cycle bounding the outer face of G. Assign charge $c_0(v) = d - 6$ to each vertex $v \in V(G)$ of degree d. Since all faces of G except for the outer one have length three and $|V(K)| = 4$, observe that G has exactly $3|V(G)| - 7$ edges. Consequently,

$$\sum_{v \in V(G)} c_0(v) = 2|E(G)| - 6|V(G)| = -14.$$

For each external vertex v of G, we redistribute the charge as follows. Let $u_1, \ldots, u_d$ be the neighbors of v in order according to the drawing of G,

where u_1vu_d is a subpath of K; since G is thick, we have $d \geq 4$. The vertex v sends $1/2$ to u_2 and u_{d-1} and 1 to each of $u_3, \ldots, u_{d-2}$. Let us remark that since all internal faces of G have length three, the vertices u_2 and u_{d-1} receive additional $1/2$ in charge from this rule being applied at u_1 and u_d, respectively.

For each vertex $x \in V(G)$, let $c(x)$ be the charge of x after the redistribution. Since no charge is created or lost, we have

$$\sum_{x \in V(G)} c(x) = \sum_{x \in V(G)} c_0(x) = -14.$$

Moreover, for each vertex $v \in V(K)$, we have

$$c(v) \geq c_0(v) - 2 \times 1/2 - (\deg v - 4) \times 1 = -3,$$

and for each internal vertex x with a neighbor in K, we have

$$c(x) \geq c_0(x) + 1 = \deg x - 5 \geq 0.$$

Thus, the sum of charges of vertices in the closed neighborhood of K is at least $-3|V(K)| = -12 > -14$. It follows that there exists a vertex $u \in V(G) \setminus V(K)$ with no external neighbor and with $c(u) = c_0(u) < 0$, i.e., $\deg u < 6$. Since G is a candidate, we have $\deg u = 5$. □

Let us now describe the reductions needed in the generation of candidates, see Figures 2 and 3 for an illustration. After contraction of an edge, we always suppress the resulting 2-faces.

- The 1-*contraction* is a contraction of an edge v_1v_2, where v_2 is an internal vertex.
- The 2-*contraction* is contraction of two edges v_1v_2 and v_3v_4, where $v_1v_2v_3v_4$ is a hollow 4-cycle (whose interior consists of two 3-faces) and $v_1, \ldots, v_4$ are internal vertices of degree five.
- The 5-*contraction* is a contraction of the five edges $v_1v, \ldots, v_5v$ incident to an internal vertex v of degree five, where $v_1, \ldots, v_5$ are also internal vertices of degree five.
- The *extended* 5-*contraction* applies in the same situation. Let $x \neq v$ be the common neighbor of v_1 and v_2. The operation consists of the 5-contraction of the edges incident with v followed by a 1-contraction of the edge between the resulting vertex and x.

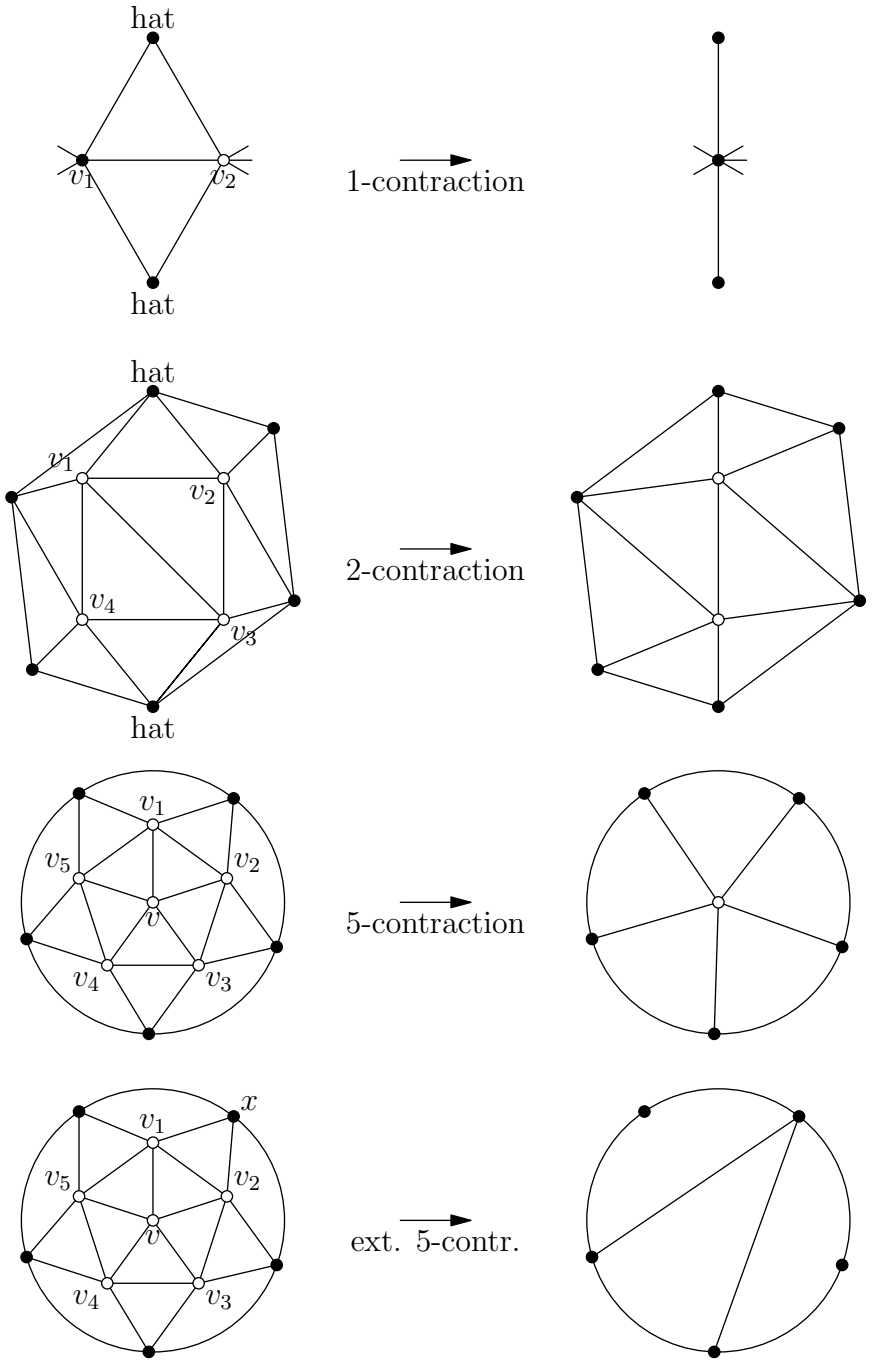


Figure 2: $\star$-contractions (1)

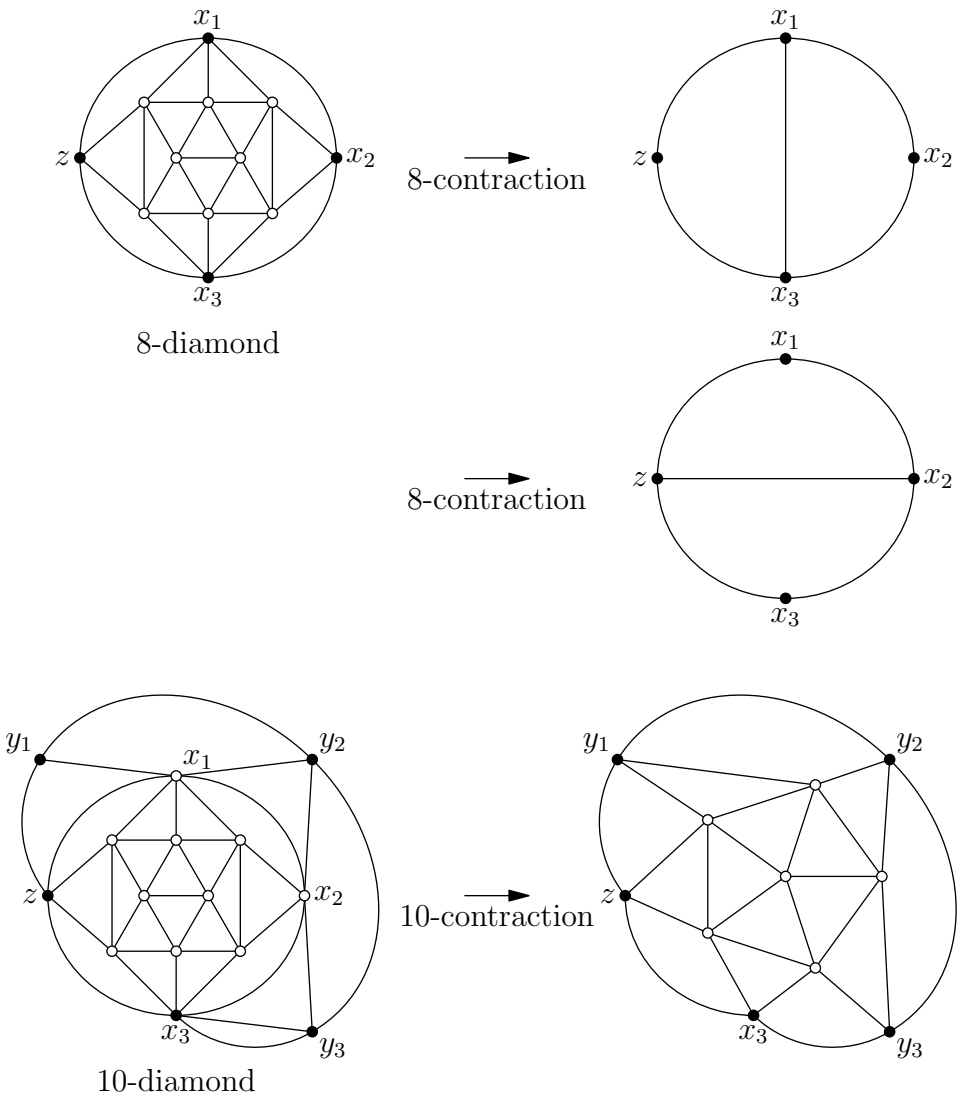


Figure 3: $\star$-contractions (2)

- Let $K = x_1x_2x_3z$ be a 4-cycle such that the subgraph H drawn in the closed disk bounded by K is as depicted in Figure 3 on the top left; in particular, all vertices of $V(H) \setminus V(K)$ are internal and have degree five. We say that H is an 8-*diamond*. The 8-*contraction* is the operation of deleting the vertices of $V(H) \setminus V(K)$ and adding either the edge x_1x_3 or the edge x_2z.

- Let $K = zy_1y_2y_3x_3$ be a 5-cycle such that the subgraph H drawn in the closed disk bounded by K consists of an 8-diamond inside the 4-cycle $zx_1x_2x_3$ and the edges of the path $y_1x_1y_2x_2y_3x_3$, as depicted in Figure 3 on the bottom left. in particular, all vertices of $V(H) \setminus (V(K) \cup \{x_1, x_2\})$ are internal and have degree five, and the vertices x_1 and x_2 are internal and have degrees 7 and 6, respectively. We say that H is a 10-*diamond*. The 10-*contraction* is the operation of deleting the vertices of $V(H) \setminus V(K)$ and replacing them by the 5-wheel of six internal vertices of degree five.

Let us remark that this set of reductions is inspired by [3]. By a $\star$-*contraction*, we mean any of these operations. We say that a $\star$-contraction is *safe* if the result of the operation is a candidate, i.e., it is internally 4-connected and all internal vertices have degree at least five. Since we are only working with simple graphs, this also implicitly contains the assumption that no parallel edges or loops (except for those eliminated by suppression of 2-faces) arise in the reduction. A $\star$-contraction is *strongly safe* if it is safe and the resulting candidate is thick.

For a 1-contraction, the *hats* are the two vertices distinct from v_1 and v_2 and incident with the faces whose boundary contains the edge v_1v_2. For a 2-contraction, the hats are the two vertices distinct from $v_1, \ldots, v_4$ and incident with the faces whose boundary contains the edge v_1v_2 or the edge v_3v_4. The safety of 1- and 2-contractions can be described in terms of the degrees of the hats as follows.

Observation 20. *Let G be a internally 5-connected ℓ-candidate. Consider a 1- or 2-contraction in G. If neither of the hats of this $\star$-contraction is an internal vertex of degree five, then the $\star$-contraction is safe. If additionally $\ell > 4$ or neither of the hats is an external vertex of degree four, then the $\star$-contraction is strongly safe.*

Proof. Let the canvas G' be created by performing the 1- or 2-contraction. Since G is internally 5-connected, observe that this $\star$-contraction cannot create parallel edges or a non-facial triangle. In the case of 1-contraction,

the vertex v of G' created by contracting the edge v_1v_2 has degree $\deg(v_1) + \deg(v_2) - 4 > \deg v_1$, and it is internal if and only if v_1 is internal. In the case of a 2-contraction, both vertices of G' created by the contractions are internal and have degree five. The degrees of vertices other than hats are unchanged, and the degree of each hat decreases by one (note that the two hats are distinct, since G does not contain non-facial triangles).

Hence, if neither of the hats is an internal vertex of degree five, then every internal vertex of G' has degree at least five. Consequently, G' is a candidate, and thus the $\star$-contraction is safe. If $\ell \geq 5$, then G' is also automatically strongly safe. For $\ell = 4$, note that every external vertex has degree at least four in G by Observation 18. If additionally neither of the hats is an external vertex of degree four, this implies that the candidate G' is also thick, and thus the $\star$-contraction is strongly safe. □

Let us now argue that with a few exceptions, a strongly safe $\star$-contraction can be performed in every internally 5-connected candidate. Let us start with a quite restrictive special case.

Lemma 21. *Let G be an internally 5-connected ℓ-candidate, not isomorphic to the 8-diamond. If G contains an internal vertex v of degree five such that all neighbors of v are internal and have degree five, then G admits a strongly safe 1-contraction, 5-contraction, or extended 5-contraction.*

Proof. Let G' be the canvas obtained from G obtained by the 5-contraction of the edges incident with v, and let w denote the vertex created by the contraction. Let $v_1, \ldots, v_5$ be the neighbors of v in order. For $i = 1, \ldots, 5$, let x_i be the common neighbor of v_i and v_{i+1} (where $v_6 = v_1$) distinct from v. Note that $x_1, \ldots, x_5$ are the neighbors of w in G' in order. For distinct $i, j \in \{1, \ldots, 5\}$, we have $x_i \neq x_j$, as otherwise G would contain a non-facial triangle. In particular, G' does not have any parallel edges. Moreover, x_i is adjacent to x_j if and only if $|i - j| \equiv \pm 1 \pmod 5$, since G is internally 5-connected. Therefore, $x_1 \ldots x_5$ is an induced cycle in G as well as in G', and consequently G' does not contain any non-facial triangles.

- Let us first consider the case that $\ell > 4$ or none of $x_1, \ldots, x_5$ is an external vertex of degree four in G. Note that $\deg_{G'} x_i = \deg_G x_i - 1$ for $i \in \{1, \ldots, 5\}$, and thus this implies that the canvas G' is thick. Hence, if the 5-contraction of the edges around v is not strongly safe, then it is not safe, and one of the vertices $x_1, \ldots, x_5$ is internal and has degree at most four in G'.

We can by symmetry assume that x_2 is an internal vertex of degree five in G, so that x_2 has degree four in G'. Let x_2' be the neighbor of x_2 distinct from x_1, v_2, v_3, and x_3; since all internal faces of G' are triangles, x_2' is adjacent to x_1 and x_3. If neither x_1 nor x_3 is an internal vertex of degree five in G, then by Observation 20, the 1-contraction of the edge x_2x_2' in G is strongly safe.

If say x_1 were an internal vertex of degree five, then since all internal faces of G are triangles, we would have $x_2'x_5 \in E(G)$. Since G is internally 5-connected and the 4-cycle $x_2'x_5x_4x_3$ is not hollow, this cycle would bound the outer face of G. But then $\ell = 4$ and x_4 would be an external vertex of degree four, contradicting the assumptions of this case.

We conclude that in this case, G admits a strongly safe 1-contraction or 5-contraction.

- Hence, we can suppose that $\ell = 4$ and say x_2 is an external vertex of degree four. Then the outer face is bounded by the cycle $x_1x_2x_3z$ for some vertex z. Note that $z \notin \{x_4, x_5\}$, since otherwise the cycle $zv_5v_4x_3$ or $zv_5v_1x_1$ would contradict the internal 5-connectivity of G. Thus, x_4 and x_5 are internal vertices.

  Let G'' be obtained from G' by contracting the edge wx_2; then the canvas G'' is obtained from G by an extended 5-contraction. Note that G'' does not contain non-facial triangles, since $x_1x_4, x_3x_5, x_1x_3 \notin E(G)$. If $x_5z \in E(G)$, then since the 4-cycle $x_5zx_3x_4$ is hollow and $\deg_G(x_4) \geq 5$, we have also $x_4z \in E(G)$, and thus G would be an 8-diamond. Hence, we can assume $x_4z \notin E(G)$, and symmetrically that $x_5z \notin E(G)$. Hence, zx_1x_5 and zx_3x_4 are not 3-faces of G, and since all internal faces of G are triangles, it follows that $\deg_G x_i \geq 6$ and $\deg_{G''} x_i \geq 4$ for $i \in \{1, 3\}$. Since $x_2x_4, x_2x_5 \in E(G'')$, we have $\deg_{G''} x_2 = 4$. Moreover, $\deg_{G''} z = \deg_G z \geq 4$ by Observation 18. If the extended 5-contraction of G to G'' is not strongly safe, it follows that G'' contains an internal vertex of degree at most four. Clearly, this vertex must be x_4 or x_5.

  Suppose that say x_5 has degree at most four in G'', and thus it has degree five in G. Let x_5' be the neighbor of x_5 distinct from x_1, v_1, v_5, and x_4. Since all internal faces of G are triangles, we have $x_5'x_1, x_5'x_4 \in E(G)$. If $x_5'x_3 \in E(G)$, then $x_5'x_1x_2x_3$ would be a non-hollow 4-cycle in G. It follows that $x_5'x_3 \notin E(G)$ and $x_5'x_4x_3$ is not a 3-face of G. Since all internal faces of G are triangles, we conclude that $\deg_G x_4 \geq 6$.

Observation 20 then implies that the 1-contraction of the edge $x_5x'_5$ in G is strongly safe.

We conclude that in this case, G admits a strongly safe 1-contraction, or 5-contraction, or extended 5-contraction.

□

Next, we consider a much less restrictive special case.

Lemma 22. *Let G be an internally 5-connected ℓ-candidate, not isomorphic to the 8-diamond. If G contains a 4-cycle $C = v_1v_2v_3v_4$ of internal vertices of degree five, then G admits a strongly safe $\star$-contraction.*

Proof. Since the candidate G is internally 5-connected, the open disk bounded by the 4-cycle C contains exactly two faces; by symmetry, we can assume that v_1v_3 is an edge drawn inside C. For $i = 1, \ldots, 4$, let x_i be the common neighbor of v_i and v_{i+1} (where $v_5 = v_1$) distinct from $v_1, \ldots, v_4$. Since $\deg v_1 = \deg v_3 = 5$ and all internal faces of G are triangles, we have $x_1x_4, x_2x_3 \in E(G)$. For $i \in \{2, 4\}$, let v'_i be the neighbor of v_i different from $v_1, \ldots, v_4, x_1, \ldots, x_4$, and similarly note that v_i is adjacent to x_{i-1} and x_i. Moreover, $C' = x_1v'_2x_2x_3v'_4x_4$ is an induced cycle, as otherwise G would not be internally 5-connected (v'_2 is not adjacent to v'_4, as otherwise by symmetry between $v'_2x_2x_3v'_4$ and $v'_2x_1x_4v'_4$, we can assume that the former 4-cycle does not bound the outer face, and thus it must be hollow and x_2 or x_3 would be an internal vertex of degree four).

Let us first consider the case that either $\ell > 4$ or none of $x_1, \ldots, x_4$ is an external vertex of degree four. By Observation 20, if the 2-contraction of v_1v_2 and v_3v_4 is not strongly safe, then x_1 or x_3 is an internal vertex of degree five; by symmetry, suppose this is the case for x_1. If x_4 is an internal vertex of degree five, then G admits a strongly safe $\star$-contraction by Lemma 21 applied to v_1 and its neighbors; hence suppose this is not the case. By Observation 20, if the 2-contraction of v_2v_3 and v_1v_4 is not strongly safe, then x_2 is an internal vertex of degree five. If x_3 is an internal vertex of degree five, then G admits a strongly safe $\star$-contraction by Lemma 21 applied to v_3 and its neighbors. Otherwise, Observation 20 implies that the 1-contraction of the edge $v_4v'_4$ is strongly safe.

Next, suppose that $\ell = 4$ and at least one of the vertices $x_1, \ldots, x_4$, say x_1, is an external vertex of degree four. Hence, the outer face of G is bounded by the cycle $x_4x_1v'_2y$ for another vertex y. Since the cycle C' is induced, we have $y \notin V(C')$, and thus the vertices x_2, x_3, and v'_4 are internal and $\deg x_4 \geq 5$. By Observation 20, if $\deg(x_3) > 5$, then the 1-contraction

of v_4v_4' is strongly safe, and if $\deg(x_2) > 5$, then the 2-contraction of v_1v_4 and v_2v_3 is strongly safe. Finally, if $\deg(x_2) = \deg(x_4) = 5$, then the claim follows from Lemma 21 applied to v_3 and its neighbors. □

We can now consider the general case of internally 5-connected candidates.

Lemma 23. *Let G be an internally 5-connected ℓ-candidate, not isomorphic to the 8-diamond. If G contains an internal vertex, then G admits a strongly safe $\star$-contraction.*

Proof. Suppose first that G has an internal vertex v of degree five, and let $v_1, \ldots, v_5$ be the neighbors of v in order. If $\ell = 4$, then by Observation 18 and Lemma 19, we can additionally choose v so that $v_1, \ldots, v_5$ are internal vertices. If three consecutive neighbors of v are internal vertices of degree five, then the claim follows by Lemma 22. Otherwise, we can by symmetry assume that neither v_1 nor v_3 is an internal vertex of degree five, and the 1-contraction of the edge vv_2 is strongly safe by Observation 20.

Hence, we can assume that all internal vertices of G have degree at least six. By Observation 18 and Lemma 19, it follows that $\ell > 4$. Hence, by Observation 20, the 1-contraction of any edge incident with an internal vertex of G is strongly safe. □

Finally, let us get rid of the assumption of internal 5-connectivity.

Lemma 24. *Every ℓ-candidate G with at least one internal vertex admits a safe $\star$-contraction.*

Proof. We can assume G is not an 8-diamond, as this 4-candidate admits a safe 8-contraction. Hence, by Lemma 23, we can assume that G is not internally 5-connected. Let G' be a minimal 4-subcandidate of G with at least one internal vertex, and let $C = x_1x_2x_3z$ be the cycle bounding the outer face of G'. Note that G' is internally 5-connected. Moreover, note that each vertex of C which is not external in G is incident with an edge not belonging to $E(G')$: If say z were internal and had no neighbors outside of G', then since every internal face of G is a triangle, the triangle x_1zx_3 would bound a face drawn outside of C, and $x_1x_2x_3$ would be a non-facial triangle of G. Hence, if the interior of C is replaced by a thick 4-candidate, then every internal vertex belonging to C has degree at least five in the resulting graph. We conclude that if G' admits a strongly safe $\star$-contraction, then this $\star$-contraction is also safe in G. Hence, by Lemma 23, we can assume G' is an 8-diamond.

For $i \in \{1, 3\}$, since $\deg_{G'} x_i = 5$, the analysis from the previous paragraph shows that if x_i is an internal vertex of G, then $\deg_G x_i > 5$. Suppose that $u \in V(C)$ is an internal vertex of G with exactly one neighbor u' outside of G', and let x and y be the common neighbors of u and u' in C. Observe that every 4-cycle containing the edge uu' is hollow, since every path of length three from u to u' in G passes through x or y and every triangle in G bounds a face. Thus, the 1-contraction of the edge uu' in G is safe. Hence, we can assume that in G, each internal vertex of C has at least two neighbors outside of G'.

Let G_1 and G_2 be the 8-contractions of G obtained by replacing the interior of C by the edges x_1x_3 and x_2z, respectively. Note that at most one of the graphs G_1 and G_2 contains a non-facial triangle, as otherwise all vertices of C would have a common neighbor and G would contain a non-facial triangle.

Suppose first that G_1 contains a non-facial triangle, and thus the vertices x_1 and x_3 have a common neighbor u distinct from x_2 and z. Note that $ux_2, uz \notin E(G)$, as otherwise G would either contain a non-facial triangle or x_2 or z would be an internal vertex with only one neighbor outside of G'. Since all internal faces of G are triangles, it follows that for $i \in \{1, 3\}$, if the vertex x_i is internal in G, then it has at least two neighbors outside of G' in addition to u, and thus it has degree at least five in G_2. Moreover, if z or x_2 is an internal vertex of G, then recall that it has at least two neighbors outside of G', and thus it also has degree at least five in G_2. It follows that the 8-contraction of G to G_2 is safe.

Similarly, if G_2 contains a non-facial triangle, then the 8-contraction to G_1 is safe. Hence, we can assume neither G_1 nor G_2 contains a non-facial triangle. If the 8-contraction neither to G_1 nor to G_2 is safe, then we can by symmetry assume that x_1 and x_2 are internal vertices of G with exactly two neighbors outside of G'. Let y_1 and y_2 be the neighbors of x_1 outside of G', and let y_2 and y_3 be the neighbors of x_2 outside of G'. Let C' be the 5-cycle $y_1y_2y_3x_3z$, and let G'' be the subgraph of G drawn in the closed disk bounded by C'. Then G'' is a 10-diamond. Since neither x_3 nor z is an internal vertex of G with all neighbors in G'', the corresponding 10-contraction is safe. □

Thus, we obtain the desired result on generation of candidates.

Corollary 25. *For every ℓ-candidate G, there exists an ℓ-candidate G' with no internal vertices such that G can be obtained from G' by a sequence of inverses to $\star$-contractions, where all intermediate canvases in this sequence are ℓ-candidates.*

Corollary 25 can be used to generate all non-isomorphic ℓ-candidates with at most n vertices in time $C_\ell(n) \cdot \mathrm{poly}(n)$, were $C_\ell(n)$ is the number of such ℓ-candidates. The basic idea is as follows: We start from the ℓ-candidates with no internal vertices, i.e., from 2-connected outerplanar graphs with ℓ vertices and all internal faces of length three (such graphs can be easily enumerated, since the graphs obtained from their duals by deleting the vertex corresponding to the outer face are exactly the subcubic trees with $\ell - 2$ vertices). We then process these ℓ-candidates as well as all ℓ-candidates generated later. For the currently processed ℓ-candidate G, we perform all possible inverses to $\star$-contractions to obtain larger ℓ-candidates. For each such ℓ-candidate G' with at most n vertices, we first check whether we have not seen it before (e.g., by storing canonical isomorphism-invariant codes of all encountered ℓ-candidates in a hash table; since we deal with plane graphs, such a canonical code can be computed in linear time), and if not, we schedule it for processing.

This procedure is guaranteed to generate all non-isomorphic ℓ-candidates with at most n vertices by Corollary 25. However, the test whether we have seen the currently considered ℓ-candidate before is somewhat time consuming and requires space linear in $C_\ell(n)$. A more efficient approach is described in [10]; briefly, for each ℓ-candidate, we canonically pick one way of obtaining it through the inverse to a $\star$-contraction, then only process it when we reach it through this particular operation. Let us mention that another advantage of this approach is that it is easy to distribute, since each candidate is processed in isolation.

We implemented this algorithm and used it to generate all non-isomorphic 4-candidates with at most 28 vertices, testing more than 10^9 graphs, see Table 1 for exact statistics[1]. None of them is rainbow-forbidding, and only the one consisting of the 4-cycle bounding the outer face with a single chord (the *diamond*) is diagonal-forbidding. This confirms Conjecture 2 for graphs with at most 28 vertices.

Interestingly, there turn out to be infinitely many bichromatic-forbidding 4-candidates. Next, we describe a construction of such candidates. To justify its validity, it is convenient to use the following result of Fisk [6].

Lemma 26. *Let G be a triangulation of an orientable surface, and let φ be a 4-coloring of G. For $i \in \{1, \ldots, 4\}$, let n_i be the number of odd degree vertices of color i. Then $n_1 \equiv \ldots \equiv n_4 \pmod 2$.*

[1] More precisely, two of the authors wrote their own implementations, one in C++, one in Python/SageMath; the latter is slower and we only used it to double-check the claim up to 25 vertices. The programs can be found at https://lidicky.name/pub/mindeg5/.

Vertices	4-candidates	bichromatic-forbidding
4	1	1
12	1	0
13	3	0
14	11	0
15	37	0
16	134	1
17	470	0
18	1 713	0
19	6 150	3
20	22 353	10
21	81 158	29
22	296 023	122
23	1 081 761	490
24	3 965 863	1 851
25	14 576 016	6 915
26	53 728 107	25 631
27	198 572 799	94 323
28	735 894 150	346 005

Table 1: Numbers of all non-isomorphic 4-candidates with up to 28 vertices, and the numbers of non-isomorphic bichromatic forbidding candidates with up to 28 vertices. No rainbow- or diagonal-forbidding candidates except for the diamond were found.

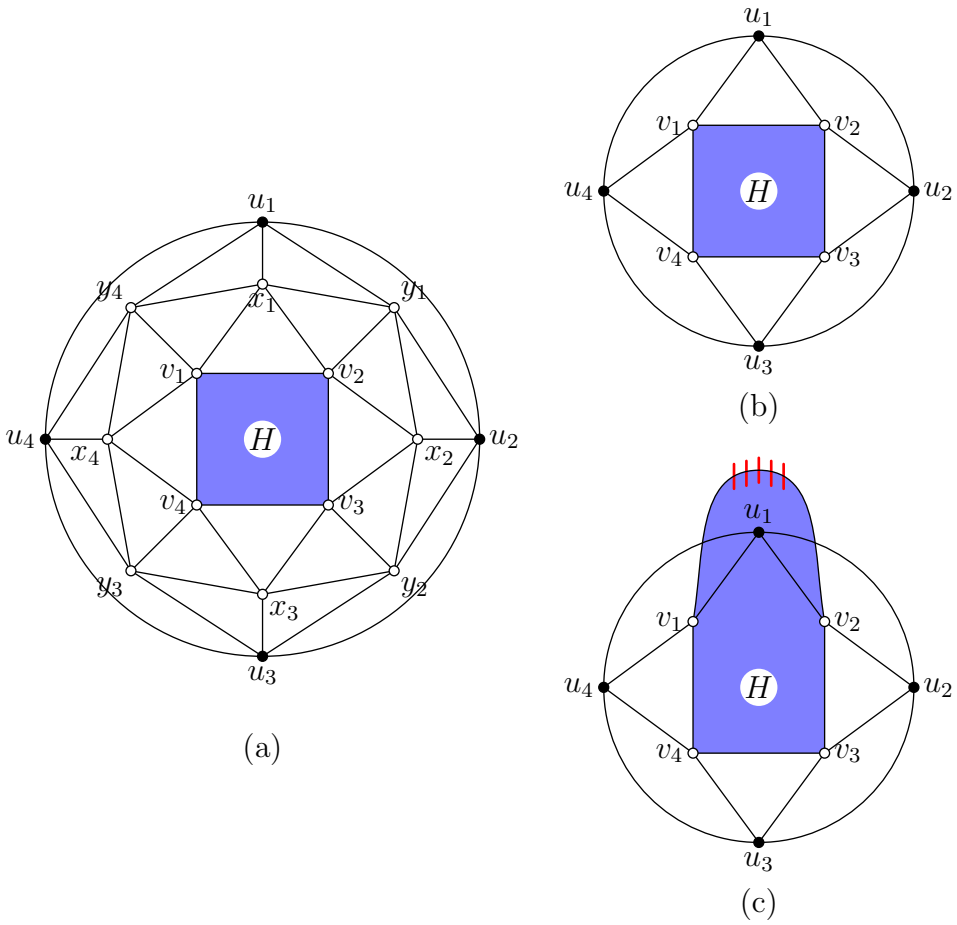


Figure 4: The constructions of bichromatic-forbidding 4-candidates.

The construction is as follows.

Lemma 27. *Let H be a plane graph with the outer face bounded by the cycle $Q = v_1v_2v_3v_4$. Let G be the plane graph obtained as follows (see Figure 4(a)): We add a cycle $x_1y_1 \dots x_4y_4$ to the outer face of H, together with edges v_ix_i, v_ix_{i-1}, and v_iy_{i-1} for $i \in \{1, \dots, 4\}$, where $x_0 = x_4$ and $y_0 = y_4$. Then, we add the 4-cycle $K = u_1 \dots u_4$ bounding the outer face, and edges u_iy_{i-1}, u_ix_i, and u_iy_i for $i \in \{1, \dots, 4\}$. The graph G is bichromatic-forbidding if and only if the graph H is bichromatic-forbidding. Moreover, if H is a 4-candidate, then G is a 4-candidate.*

Proof. Suppose first that G is not bichromatic-forbidding, and thus G has a 4-coloring φ such that $\varphi(u_1) = \varphi(u_3) = 1$ and $\varphi(u_2) = \varphi(u_4) = 2$. We claim that φ uses only two colors on Q, and thus the restriction of φ to H shows that H is not bichromatic-forbidding. For contradiction, suppose that φ uses at least three colors on Q, and thus by symmetry, we can assume that $\varphi(v_1) \neq \varphi(v_3)$.

Let G' be the triangulation of the plane obtained from $G - (V(H) \setminus V(K)) - (E(H) \setminus E(K))$ by adding the edge v_1v_3 and a vertex z adjacent to $u_1, \dots, u_4$, and let φ' be the 4-coloring of G' obtained from the restriction of φ to $V(G) \setminus (V(H) \setminus V(K))$ by letting $\varphi'(z) = 3$. Let $M = \{v_1, \dots, v_4, x_1, \dots, x_4, y_1, \dots, y_4\}$ and for $c \in \{1, \dots, 4\}$, let m_c be the number of vertices in M of color c. Note that M is covered by four triangles in $G'[M]$, namely $v_ix_{i-1}y_{i-1}$ for $i \in \{1, \dots, 4\}$, and thus $m_1, \dots, m_4 \leq 4$. Moreover, since $\varphi'(u_1) = \varphi'(u_3) = 1$, the color 1 can only be used on M on the triangles $v_1v_4x_4$ and $v_2v_3x_2$, and thus $m_1 \leq 2$, and symmetrically $m_2 \leq 2$. Since $m_1 + \dots + m_4 = |M| = 12$, this is only possible if $m_1 = m_2 = 2$ and $m_3 = m_4 = 4$.

Note that the odd degree vertices of G' are exactly those in $M' = M \setminus \{v_1, v_3\}$. Each of the two colors that do not appear on v_1 and v_3 is used an even number of times on M' (twice or four times). By Lemma 26, the remaining two colors $\varphi'(v_1)$ and $\varphi'(v_3)$ also have to be used even number of times on M'. However, then the color $\varphi'(v_1)$ would be used odd number of times on M, which is a contradiction, since $m_{\varphi'(v_1)} \in \{2, 4\}$.

Conversely, suppose that H is not bichromatic-forbidding, and thus H has a 4-coloring ψ such that $\psi(v_1) = \psi(v_3) = 1$ and $\psi(v_2) = \psi(v_4) = 2$. Let us extend ψ by letting $\psi(x_1) = \dots = \psi(x_4) = 3$, $\psi(y_1) = \dots = \psi(y_4) = 4$, $\psi(u_1) = \psi(u_3) = 1$ and $\psi(u_2) = \psi(u_4) = 2$. This gives a 4-coloring of G which shows that G is not bichromatic-forbidding.

Therefore, the graph G is bichromatic-forbidding if and only if the graph H is. If H is a 4-candidate, then note that the construction of G does not

create non-facial triangles, and clearly all vertices of M have degree at least five in G; therefore, G is a 4-candidate as well. □

According to Lemma 27, we can start with the diamond (a trivial bichromatic-forbidding 4-candidate) and iterate the construction described in the statement of the lemma to obtain larger and larger bichromatic-forbidding 4-candidates.

A simpler construction is given in the following lemma.

Lemma 28. *Let H be a 4-candidate with the outer face bounded by an induced cycle $Q = v_1v_2v_3v_4$. Let G be a canvas obtained from H by adding a cycle $K = u_1u_2u_3u_4$ bounding the outer face and the edges u_iv_i and u_iv_{i+1} for $i \in \{1, \ldots, 4\}$, where $v_5 = v_1$ (see Figure 4(b)). Then G is a 4-candidate, and it is bichromatic-forbidding if and only if H is.*

Proof. Since Q is an induced cycle in H, all its vertices have degree at least three in H and at least five in G. It follows that G is a candidate.

Suppose that G is not bichromatic-forbidding, and thus G has a 4-coloring φ such that $\varphi(u_1) = \varphi(u_3) = 1$ and $\varphi(u_2) = \varphi(u_4) = 2$. Then φ can only use colors 3 and 4 on the 4-cycle Q, and thus the restriction of φ to H shows that H is not bichromatic-forbidding.

Conversely, if H is not bichromatic-forbidding, then H has a 4-coloring ψ such that $\psi(v_1) = \psi(v_3) = 3$ and $\psi(v_2) = \psi(v_4) = 4$. We can extend ψ to a 4-coloring of G by letting $\psi(u_1) = \psi(u_3) = 1$ and $\psi(u_2) = \psi(u_4) = 2$, thus showing that G is not bichromatic-forbidding.

Therefore, G is bichromatic-forbidding if and only if H is. □

Since Lemma 28 requires Q to be an induced cycle, it cannot be applied to the diamond directly. However, it can be applied to the 4-candidates arising from Lemma 27. Note that Lemma 27 produces 4-candidates with all external vertices of degree five and Lemma 28 ones with all external vertices of degree four. There is also the following variant of Lemma 28 producing 4-candidates with three external vertices of degree four and one of degree at least seven.

Lemma 29. *Let H be a 4-candidate with the outer face bounded by an induced cycle $Q = v_1v_2v_3v_4$, and let $v_1v_2u_1$ be the triangle bounding the other face incident with the edge v_1v_2. Let G be a canvas obtained from $H - v_1v_2$ by adding a cycle $K = u_1u_2u_3u_4$ bounding the outer face and the edges u_iv_i and u_iv_{i+1} for $i \in \{2, 3, 4\}$, where $v_5 = v_1$ (see Figure 4(c)). If $\deg_H v_1, \deg_H v_2 \geq 5$, then G is a 4-candidate. Moreover, G is bichromatic-forbidding if and only if H is.*

Proof. If $\deg_H v_1 \geq 5$ and $\deg_H v_2 \geq 5$, then all vertices of Q have degree at least five in G. From this, it follows that G is a candidate.

Suppose that G is not bichromatic-forbidding, and thus G has a 4-coloring φ such that $\varphi(u_1) = \varphi(u_3) = 1$ and $\varphi(u_2) = \varphi(u_4) = 2$. Then φ can only use colors 3 and 4 on the path $Q - v_1v_2$. This implies that $\varphi(v_1) \neq \varphi(v_2)$, and thus the restriction of φ to $V(H)$ is a proper 4-coloring of H. This 4-coloring shows that H is not bichromatic-forbidding.

Conversely, if H is not bichromatic-forbidding, then H has a 4-coloring ψ such that $\psi(v_1) = \psi(v_3) = 3$ and $\psi(v_2) = \psi(v_4) = 4$. By symmetry, we can assume that $\psi(u_1) = 1$. We can extend ψ to a 4-coloring of G by letting $\psi(u_3) = 1$ and $\psi(u_2) = \psi(u_4) = 2$, thus showing that G is not bichromatic-forbidding.

Therefore, G is bichromatic-forbidding if and only if H is. □

The aforementioned enumeration of 4-candidates with at most 28 vertices supports the following conjecture.

Conjecture 30. *Every bichromatic-forbidding 4-candidate can be obtained from the diamond by a sequence of the operations described in Lemmas 27, 28, and 29.*

3 Generating rainbow-forbidding weak 4-candidates

A canvas G is a *weak ℓ-candidate* if its outer face is bounded by an ℓ-cycle and all triangles in G bound faces. Note that this in particular implies that all internal vertices of G have degree at least four. It is easy to see that to enumerate non-4-colorable graphs in $\mathcal{C}_0$, it suffices to enumerate the rainbow-forbidding weak 4-candidates. To do so, we fundamentally use the reduction discussed in the introduction, deletion of an internal vertex v of degree four followed by identification of two non-adjacent neighbors of v. An issue here is that this might create a non-facial triangle, leading to a somewhat more involved argument presented in this section. Let us remark that the same procedure can be used to generate diagonal-forbidding weak 4-candidates; thus, we say that a plane graph H with the outer face bounded by a 4-cycle is *restrictive* if it is rainbow- or diagonal-forbidding. The *kind* of H is `r` if H is rainbow-forbidding and `d` if H is diagonal-forbidding.

Let G be a restrictive weak 4-candidate and let $P = uvw$ be a path in G, where

- v is an internal vertex of degree four and u, x, w, y are its neighbors in order according to the plane drawing of G,

- at least one of the vertices u and w is internal, and
- $uw \notin E(G)$ and x, y, and v are the only common neighbors of u and w.

Let G/P be the canvas obtained by contracting the edges of P and suppressing the 2-faces; by the assumptions, G/P does not have loops or parallel edges. We say that a vertex z of G/P is *triangle-isolated* if there exists a triangle T in G/P such that the open disk bounded by T contains z. For a non-negative integer m, we say that the path P is *m-contractible* if it satisfies the conditions described above and the open disk bounded by any triangle in G/P contains at most m vertices. Since every triangle in G is facial, observe that all triangle-isolated vertices in G/P are contained in open disks bounded by at most two triangles, corresponding to 5-cycles in G passing through P and drawn on the opposite sides of P. Hence, if the path P is m-contractible, then G/P contains at most $2m$ triangle-isolated vertices.

Observation 31. *Let G be a weak 4-candidate, let $P = uvw$ be an m-contractible path in G for an integer $m \geq 0$, and let F be the graph obtained from G/P by deleting all triangle-isolated vertices. If G is restrictive, then F is restrictive as well, and F has the same kind as G.*

Proof. It suffices to show that every 4-coloring φ of F extends to a 4-coloring of G. First, for every non-facial triangle T in G/P, we can extend φ to the subgraph of G/P drawn inside T: The subgraph has a 4-coloring by the Four Color Theorem, and by permuting the colors, we can modify it to match φ on T. Thus, φ extends to a 4-coloring of G/P. Next, we extend φ to G: We give u and w the color of the vertex resulting from the contraction of P, and then extend the coloring to v greedily. □

It will be convenient to consider a variant of this operation: A *bidiamond* consists of two adjacent internal vertices u and v of degree four. Let x and y be the common neighbors of u and v, and let u' and v' be the neighbors of u and v, respectively, not in $\{u, v, x, y\}$. If at least one of x and y is internal and there is no path of length at most three from x to y in $G - \{u', u, v, v'\}$, then we say that the bidiamond is *contractible*. The *contraction* of the bidiamond is the canvas obtained from $G - \{u, v\}$ by identifying x with y and suppressing the 2-faces. Observe that the resulting canvas is a restrictive weak 4-candidate of the same kind as G. According to the following lemma, even non-contractible bidiamonds are useful.

Lemma 32. *Let G be a restrictive triangulated weak 4-candidate, and suppose that vertices u and v form a bidiamond in G. Then either this bidiamond is contractible, or G contains a 1-contractible path.*

Proof. Let x and y be the common neighbors of u and v, and let u' and v' be the neighbors of u and v, respectively, not in $\{u, v, x, y\}$. We can assume that the path $u'uv$ is not 1-contractible, and in particular there exists a path of length at most three between u' and v in $G - \{u, x, y\}$. The vertices u' and v' are non-adjacent, since G does not contain separating triangles. Therefore, u' and v' have a common neighbor $z \notin \{x, y\}$.

By symmetry, we can assume that the open disk Λ_1 bounded by the 5-cycle $u'uvv'z$ contains x; and in particular, x is an internal vertex. Note that since G does not contain non-facial triangles, z is not adjacent to y. Let z' be a common neighbor of u' and v' such that the open disk Λ_2 bounded by the 5-cycle $u'uvv'z'$ contains the face bounded by the triangle uvy; note that $z' = y$ is possible. Subject to these properties, choose the vertices z and z' so that the disks Λ_1 and Λ_2 are as large as possible.

We can assume that the bidiamond $\{u, v\}$ is not contractible, and thus there exists a path R of length at most three from x to y in $G - \{u', u, v, v'\}$. By planarity, R intersects the path $u'zv'$, and since $zy \notin E(G)$, we conclude that $R = xzy'y$ for a vertex y'. Since all triangles is G bound faces and $xz \in E(G)$, note that x is the only vertex contained in Λ_1. Moreover, observe that either $z' = y$ or $z' = y'$, and thus only the vertex y can be contained in Λ_2. By the maximality of Λ_1 and Λ_2, it follows that only x and possibly y is triangle-isolated in $G/u'uv$, and thus the path $u'uv$ is 1-contractible. □

We also need the following observations.

Observation 33. *Let G be a plane graph and let P be a path of length at most two contained in the boundary of a face of G. Then every precoloring of $G[V(P)]$ extends to a 4-coloring of G.*

Proof. We can assume that P has length two, as otherwise we can extend P by another vertex; let $P = v_1v_2v_3$. Let ψ be any precoloring of $G[V(P)]$. Add the edge v_1v_3 to G, and if $\psi(v_1) = \psi(v_3)$, contract this edge. The resulting planar graph G' is 4-colorable by the Four Color Theorem, and we can permute the colors in this 4-coloring to match ψ. Thus, the 4-coloring of G' corresponds to a 4-coloring of G extending ψ. □

Observation 34. *Let G be a plane graph and let P be a path of length three contained in the boundary of a face of G. Then every precoloring of $G[V(P)]$ giving the ends of P the same color extends to a 4-coloring of G.*

Proof. Let $P = v_1v_2v_3v_4$ and let ψ be any precoloring of $G[V(P)]$ such that $\psi(v_1) = \psi(v_4)$. Since ψ exists, we have $v_1v_4 \notin E(G)$. Identify the vertices v_1 and v_4, turning P into a triangle. The resulting planar graph G' is 4-colorable by the Four Color Theorem, and we can permute the colors in this 4-coloring to match ψ. Thus, the 4-coloring of G' corresponds to a 4-coloring of G extending ψ. □

We say that a canvas is *trivial* if it has at most one internal vertex. A restrictive weak 4-candidate G is *deepest* if it is non-trivial and no proper subcanvas of G is a non-trivial restrictive weak 4-candidate. Lemma 32 makes the following observation useful even when G is a subcanvas of a larger weak 4-candidate.

Lemma 35. *Let G be a deepest restrictive weak 4-candidate with the outer face bounded by the cycle $K = x_1x_2x_3x_4$. If x_1 and x_3 have a common neighbor $v \notin \{x_2, x_4\}$, then G contains a bidiamond.*

Proof. For $i \in \{1, 2\}$, let K_i be the 4-cycle $x_{2i-1}x_2vx_4$ and let G_i be the subcanvas of G bounded by K_i. Since G is non-trivial and all triangles in G bound faces, G contains at most one of the edges x_1v and x_3v. We claim that both G_1 and G_2 are trivial, and thus G contains a bidiamond. For contradiction, suppose that say G_1 is non-trivial. Clearly G_1 is a weak 4-candidate, and since G is deepest, G_1 is not restrictive.

Let φ be a coloring of K that does not extend to a 4-coloring of G, such that the type of φ is rainbow or diagonal. By Observation 33, the restriction of φ to the path $x_2x_3x_4$ extends to a 4-coloring φ_2 of G_2. Let ψ_1 be the 4-coloring of K_1 matching φ on x_2, x_1, and x_4, and matching φ_2 on v. Since φ does not extend to a 4-coloring of G, the coloring ψ_1 does not extend to a 4-coloring of G_1. Since G_1 is not restrictive, we can assume that $\varphi(x_1) = \psi_1(v) = 1$ and $\varphi(x_2) = \varphi(x_4) = 2$. Since the type of φ is not bichromatic, we can assume that $\varphi(x_3) = 3$.

Observe that Corollary 17 implies that there exists a color $c \in \{3, 4\}$ such that G_2 has a 4-coloring φ_2' matching φ on the path $x_2x_3x_4$ and such that $\varphi_2'(v) = c$. Since G_1 is not restrictive, it has has a 4-coloring φ_1 matching φ on the path $x_2x_1x_4$ and such that $\varphi_1(v) = c$. The combination of φ_2' and φ_1 is a 4-coloring of G extending φ, which is a contradiction. □

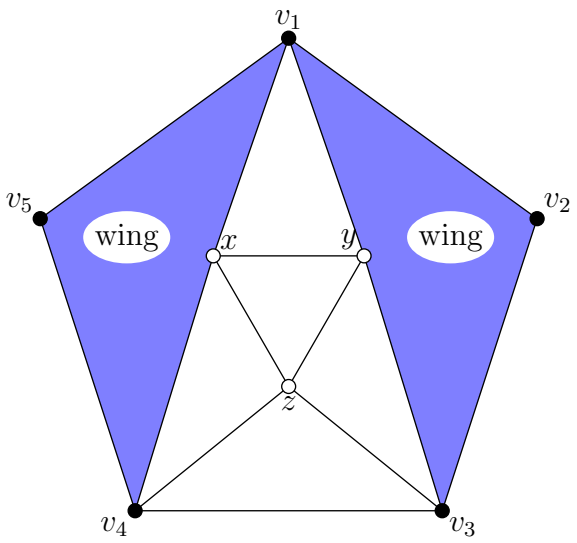


Figure 5: A troublesome pentagon.

Next, we need to consider properties of two special configurations. Let G be a weak 4-candidate. A *troublesome pentagon* in G is a subcanvas H of G with the outer face bounded by a 5-cycle $Q = v_1 \dots v_5$ containing a triangle xyz of internal vertices such that x is adjacent to v_1 and v_4, y is adjacent to v_1 and v_3, and z is adjacent to v_3 and v_4; see Figure 5. The vertex z is the *center* of the pentagon. The subcanvases of H bounded by the cycles $v_1xv_4v_5$ and $v_1v_2v_3y$ are the *wings* of the troublesome pentagon. We say that a vertex w of G is *potentially useful* if there exists a restrictive induced subgraph F of G with the same outer face as G such that $F - w$ is not restrictive.

Lemma 36. *Let G be a restrictive weak 4-candidate and let H be a troublesome pentagon in G. If neither of the wings of H is restrictive, then the center of H is not potentially useful.*

Proof. Let K be the 4-cycle bounding the outer face of G. Let $Q = v_1 \dots v_5$ and xyz be as in the definition of a troublesome pentagon and let H_1 and H_2 be the wings, where $v_2 \in V(H_2)$. Suppose for a contradiction that z is potentially useful, and thus there exists a restrictive induced subgraph F of G with the outer face bounded by K such that $F - z$ is not restrictive. Let ψ be a precoloring of K (of rainbow or diagonal type) that does not extend to a 4-coloring of F, but does extend to a 4-coloring of $F - z$.

Let $F' = (F - z) \cup Q$. Note that F' is obtained from $F - z$ by adding the vertices of $V(Q) \setminus V(F)$ and the incident edges of Q, and thus $F - z$ is an induced subgraph of F' and all vertices of $V(F') \setminus V(F)$ have degree two in F'. Hence, ψ also extends to a 4-coloring of F'. Let F'' be the subgraph of F' obtained by deleting the vertices and edges drawn in the open disk bounded by Q.

Consider any 4-coloring φ of F'' extending ψ. Since ψ does not extend to a 4-coloring of F, the coloring φ does not extend to a 4-coloring of H. We claim that this implies the following:

($\star$) Every 4-coloring φ of F'' extending ψ satisfies either $\varphi(v_1) = \varphi(v_4)$ and $\varphi(v_3) = \varphi(v_5)$, or $\varphi(v_1) = \varphi(v_3)$ and $\varphi(v_2) = \varphi(v_4)$.

Indeed, if $\varphi(v_1) \notin \{\varphi(v_3), \varphi(v_4)\}$, then we could give x the color $\varphi(v_3)$, y the color $\varphi(v_4)$, extend the coloring to z, and extend the coloring to H_1 and H_2 using the assumption that they are not restrictive, which is a contradiction. Hence, suppose that $\varphi(v_1) = \varphi(v_4)$ (the case $\varphi(v_1) = \varphi(v_3)$ is symmetric). Then we can color x by $\varphi(v_3)$ and extend the 4-coloring to z, y, and H_2. Since φ does not extend to H, this coloring does not extend to H_1. It follows that the subcanvas H_1 is bichromatic-forbidding and the coloring of its boundary 4-cycle given by φ and the assignment of the color $\varphi(v_3)$ to x uses only two colors, i.e., $\varphi(v_5) = \varphi(v_3)$. This confirms the conclusion of ($\star$).

Since ψ extends to a 4-coloring of $F - z$, it also extends to a 4-coloring φ of F''. By ($\star$), we can assume that $\varphi(v_1) = \varphi(v_4) = 1$ and $\varphi(v_3) = \varphi(v_5) = 2$. For distinct colors $a, b \in \{1, \ldots, 4\}$, let $C_{a,b}$ be the subgraph of F'' induced by the vertices whose colors according to φ are a or b.

If v_1 and v_4 are in the same component of $C_{1,3}$, then by planarity, there exists $i \in \{3, 5\}$ such that the component C_1 of $C_{2,4}$ containing v_i does not contain v_{8-i} and does not intersect K. We can exchange the colors 2 and 4 on C_1 to obtain a 4-coloring of F'' violating ($\star$), which is a contradiction. Therefore, v_1 and v_4 are not in the same component of $C_{1,3}$, and similarly, they are not in the same component of $C_{1,4}$.

If there exists $i \in \{1, 4\}$ such that the component C_2 of $C_{1,3}$ or $C_{1,4}$ containing v_i is disjoint from K, we can exchange the colors on C_2, again obtaining a 4-coloring of F'' violating ($\star$). Thus, we can assume that all such components intersect K. Since v_1 and v_4 are not in the same component of $C_{1,3}$ or $C_{1,4}$, for both $a \in \{3, 4\}$, there exist distinct non-adjacent vertices $s_a, t_a \in V(K)$ such that $\psi(s_a), \psi(t_a) \in \{1, a\}$ and all other vertices of K have colors different from 1 and a.

If $\{s_3, t_3\} = \{s_4, t_4\}$, this implies that $\psi(s_3) = \psi(t_3) = 1$ and ψ assigns the color 2 to the remaining two vertices of K. If $\{s_3, t_3\} \neq \{s_4, t_4\}$, then $\psi(s_3) = \psi(t_3) = 3$ and $\psi(s_4) = \psi(t_4) = 4$. In either case, the type of ψ were bichromatic, which is a contradiction. □

Let G be a weak 4-candidate. An *amoeba* in G is a subcanvas H of G with the outer face bounded by a 4-cycle $Q = v_1 \dots v_4$, where H contains a triangle wyz of internal vertices, v_1 is adjacent to w and y, v_2 is adjacent to w and z, and v_3 is adjacent to z, and moreover y and v_3 have a common neighbor $x \neq z$, where either x is an internal vertex of H or $x = v_4$; see bottom right of Figure 6. The subcanvas of G bounded by the 4-cycle $C = xyzv_3$ is the *eye* of the amoeba, and the subgraph of G resulting from deleting all vertices and edges drawn in the open disk bounded by C is said to be obtained from G by *blinding* the amoeba. The amoeba is *proper* if its eye has at least one internal vertex. The eye of H is irrelevant for the restrictive precoloring extension in G, in the following sense.

Lemma 37. *Let G be a restrictive weak 4-candidate, let ψ be a precoloring of the 4-cycle K bounding the outer face of G that does not extend to a precoloring of G, and let H be an amoeba in G. Then ψ does not extend to a 4-coloring of the subgraph G' of G obtained by blinding the amoeba H.*

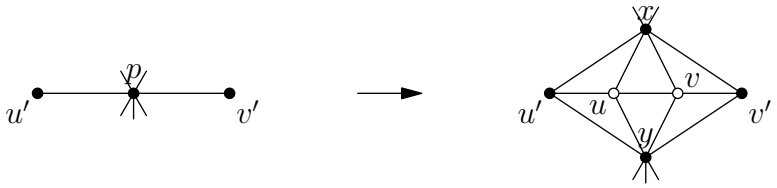
Proof. Let $Q = v_1 \dots v_4$, w, x, y, and z be as in the definition of an amoeba. Suppose for a contradiction that ψ extends to a 4-coloring φ of G'.

If $\varphi(v_1) \neq \varphi(v_3)$, then we let φ_0 be the restriction of φ to the vertices of G which are not internal in H and we show that φ_0 can be extended to a 4-coloring of G, obtaining a contradiction: We extend φ_0 to z by letting $\varphi_0(z) = \varphi(v_1)$. By Observation 34, φ_0 further extends to the subcanvas of G bounded by the 5-cycle $zv_3v_4v_1y$. Since $\varphi_0(v_1) = \varphi_0(z)$, we can finally extend this 4-coloring to w as well.

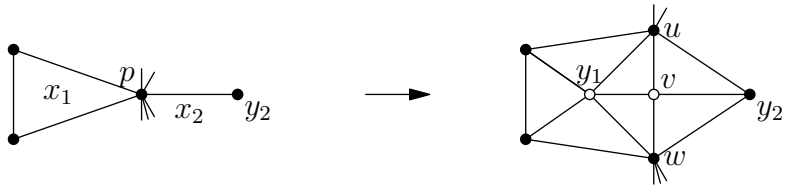
Therefore, we have $\varphi(v_1) = \varphi(v_3)$, and thus $\varphi(z) \neq \varphi(v_1)$, and consequently $\varphi(y) = \varphi(v_2)$. Let φ_1 be the restriction of φ obtained by uncoloring the vertices z and w. By Observation 33, φ_1 can be extended to the eye of H, and since $\varphi_1(y) = \varphi_1(v_2)$, we can finally extend φ_1 to w as well. This is again a contradiction. □

Finally, we are ready to state the result showing that every non-trivial restrictive weak 4-candidate contains a reducible configuration.

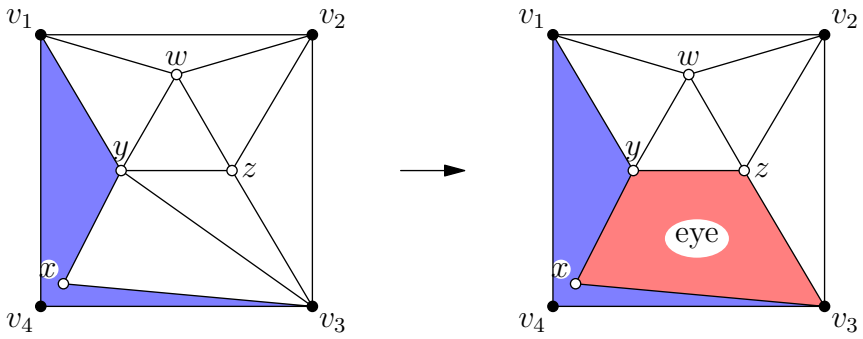
Theorem 38. *If Conjecture 2 holds, then every non-trivial restrictive weak 4-candidate G contains a contractible bidiamond, a 1-contractible path, or a proper amoeba.*



decontraction of a bidiamond



decontraction of a 1-contractible path



amoeba expansion

Figure 6: Configurations from Theorem 38 and the corresponding expansions.

Proof. By Lemma 32, we can assume that G does not contain a bidiamond. Let H be a minimal non-trivial restrictive subcanvas of G; note that H is deepest. Let Q be the 4-cycle bounding the outer face of H. Since H is non-trivial and all triangles in G are facial, the cycle Q is induced. By Lemmas 32 and 35, we can assume that no internal vertex of H has non-adjacent neighbors in Q.

Suppose now that there exists an induced path $S = v_1vv_3w$ in H such that $v_1, w \in V(Q)$, $v, v_3 \notin V(Q)$, and v has degree four. Let $v_1v_2v_3v_4$ be the cycle formed by the neighbors of v. Note that $\{v_2, v_4\} \not\subseteq V(Q)$, since v does not have two non-adjacent neighbors in Q; by symmetry, we can assume that $v_4 \notin V(Q)$. Let $Q = v_1u_1wu_2$, where v_4 is drawn in the open disk bounded by the 5-cycle $v_1vv_3wu_2$. Since v_4 does not have two non-adjacent neighbors in Q, it is not adjacent to w. Similarly, if $v_2 \neq u_1$, then $v_2w \notin E(G)$. We can assume that the path v_2vv_4 is not 1-contractible in G, and thus there exists an induced path R of length at most three in $G - \{v, v_1, v_3\}$ between v_2 and v_4. This is not possible if $v_2 \neq u_1$, since then $v_4w, v_2w, u_1u_2 \notin E(G)$. Therefore, $v_2 = u_1$.

Suppose that v_4 is adjacent to u_2. Note that v_3 is not adjacent to v_2, as otherwise v and v_4 would form a bidiamond. Consider any 4-coloring ψ of Q whose type is rainbow or diagonal. By symmetry, we can assume that $\psi(v_1) \neq \psi(w)$. We can color v_3 by the color $\psi(v_1)$ and by Observation 33, extend the 4-coloring to the subcanvas drawn in the closed disk bounded by the 4-cycle $wv_3v_4u_2$. Finally, we can greedily extend the coloring to v, since two of its neighbors (v_1 and v_3) have the same color. Therefore, any such 4-coloring of Q extends to a 4-coloring H, which is a contradiction since H is restrictive.

Hence, we can assume that $v_4u_2 \notin E(G)$. Let us now continue the discussion of the path R showing that the path v_2vv_4 is not 1-contractible. Since v_4 is also not adjacent to w, the neighbor x of v_4 in R is contained in $V(H) \setminus V(Q)$. Since Q is an induced cycle, it follows that $R = v_4xwv_2$. Therefore, H is an amoeba. Since the path R shows that v_2vv_4 is not 1-contractible, the open disk bounded by the 5-cycle v_2vv_4xw must contain at least two vertices, and thus the eye of the amoeba H has at least one internal vertex. It follows that the amoeba H is proper, and thus the conclusion of the Lemma holds. In conclusion,

($\star$) there is no induced path in H of length at most three with ends in Q containing a vertex in $V(H) \setminus V(Q)$ of degree four. In particular, every induced (≤ 5)-cycle C in G such that C contains a vertex in $V(H) \setminus V(Q)$ of degree four satisfies $C \subseteq H$.

Recall that the canvas H is restrictive, and let H' be a minimal induced subgraph of H such that $Q \subseteq H'$ and H' is restrictive. In particular, all internal vertices of H' are potentially useful in H. Since H is non-trivial and every triangle in H bounds a face, observe that H' is non-trivial as well. By Lemma 13, there exists an internal vertex $v \in V(H')$ of degree four such that all incident faces of H' have length three. Thus, v also has degree four in H.

Let $v_1v_2v_3v_4$ be the cycle formed by the neighbors of v; since each triangle in G bounds a face, this cycle is induced. As we have observed, v does not have two non-adjacent neighbors in Q, and thus $\{v_1, v_3\} \not\subseteq V(Q)$ and $\{v_2, v_4\} \not\subseteq V(Q)$. In particular, v_1 or v_3 is an internal vertex, and v_2 or v_4 is an internal vertex of H.

We can assume that the path $P_1 = v_1vv_3$ is not 1-contractible, as otherwise the conclusion of the Lemma holds. Thus, $G - \{v, v_2, v_4\}$ contains an induced path R_1 of length at most three from v_1 to v_3 such that either R_1 has length two, or R_1 has length three and the open disk bounded by the triangle T_1 in G/P_1 corresponding to R_1 contains at least two vertices. By symmetry, we can assume that v_4 is contained in the open disk Λ_1 bounded by the cycle $P_1 \cup R_1$ in G.

Similarly, we can assume that the path $P_2 = v_2vv_4$ is not 1-contractible, and thus $G-\{v, v_1, v_3\}$ contains an induced path R_2 of length at most three from v_2 to v_4 such that either R_2 has length two, or R_2 has length three and the open disk bounded by the triangle T_2 in G/P_2 corresponding to R_2 contains at least two vertices. By symmetry, we can assume that v_3 is contained in the open disk Λ_2 bounded by the cycle $P_2 \cup R_2$ in G.

By $(\star)$, we have $R_1 \cup R_2 \subseteq H$. By planarity, the paths R_1 and R_2 intersect in (at least) a vertex $z \in V(H) \setminus \{v, v_1, \ldots, v_4\}$. Since each triangle in G bounds a face, z is not adjacent to both v_1 in R_1 and v_2 in R_2. By symmetry, we can assume that z is not adjacent to v_1 in R_1, and thus the path R_1 has length three and $R_1 = v_1z_1zv_3$ for a vertex z_1.

Suppose now that z is adjacent to v_2 in R_2. If R_2 has length two, then z is also adjacent to v_4, and since all triangles in G bound faces, it follows that $\deg v_3 = 4$; however, then v and v_3 would form a bidiamond. Therefore, R_2 has length three, and thus the subcanvas H'' of H drawn in the closed disk bounded by the 4-cycle $v_1z_1zv_2$ is an amoeba. Moreover, since the open disk bounded by T_2 contains at least two vertices, the eye of this amoeba contains at least one internal vertex. Therefore, the amoeba H'' is proper, and thus the conclusion of the Lemma holds.

Hence, we can assume that z is not adjacent to v_2 in R_2, and thus the path R_2 has length three and $R_2 = v_2z_2zv_4$ for a vertex z_2. If $v_1z_2 \in$

$E(G)$, then note that $v_1v_2z_2$ is a triangle bounding a face. Observe that the subcanvas of G drawn in the closed disk bounded by the 4-cycle $zv_4v_1z_2$ is an amoeba. Moreover, since the open disk bounded by T_2 contains at least two vertices, this amoeba is proper, and thus the conclusion of the Lemma holds. Therefore, we can assume that $v_1z_2 \notin E(G)$, and symmetrically $v_2z_1 \notin E(G)$.

Let F be the subcanvas of H drawn in the closed disk bounded by the 5-cycle $v_1v_2z_2zz_1$. Then F is a troublesome pentagon with center v. Since v is potentially useful in H, a wing of F is restrictive. Thus, by symmetry, we can assume that the subcanvas H_1 of H drawn in the closed disk bounded by the cycle $Q_1 = v_1v_4zz_1$ is restrictive.

By the minimality of H, the subcanvas H_1 is trivial. Since the open disk bounded by T_1 contains at least two vertices, H_1 has exactly one internal vertex y and $\deg y = 4$. Since the open disk bounded by T_2 contains at least two vertices, we have $v_3z_2 \notin E(G)$. Since $v_2z_1 \notin E(G)$, it follows that the distance between z_1 and v_4 in $G - \{v_1, y, z\}$ is greater than three, and thus the path v_4yz_1 is 0-contractible. Therefore, the conclusion of the theorem holds. □

Let F be a weak 4-candidate. We say that a weak 4-candidate G is an *expansion* of F if G is obtained from F by one of the following operations, see Figure 6:

- Decontraction of a bidiamond: Choose a path $u'pv'$ in F. Double the edges $u'p$ and pv', letting f_1 and f_2 be the resulting 2-faces. Split p into two vertices x and y along a simple curve from f_1 to f_2, and add new vertices u and v and edges ux, $u'u$, uy, uv, vx, vv', and vy to the resulting face.

- Decontraction of a 1-contractible path: Choose a vertex p of F. For $i \in \{1, 2\}$, either

  - choose an edge $x_i = py_i$ incident with p, double it and let f_i be the resulting 2-face, or
  - choose a 3-face x_i incident with p, add a vertex y_i of degree four inside x_i adjacent to all the vertices of the boundary of x_i and with the edge x_ip having multiplicity two, and let f_i be the 2-face bounded by this double edge,

  so that $x_i \neq x_{3-i}$, and x_i is not incident with x_{3-i} in the case that x_i is an edge and x_{3-i} is a 3-face. Split p into two vertices u and w along

a simple curve from f_1 to f_2, and add a vertex v of degree four to the resulting 4-face bounded by the cycle uy_1wy_2.

- Amoeba expansion: Choose a non-proper amoeba in F and replace its eye by any weak 4-candidate with at least one internal vertex.

With this definition, Theorem 38 can be reformulated as follows.

Corollary 39. *If Conjecture 2 holds, then for every non-trivial restrictive weak 4-candidate G, there exists a restrictive weak 4-candidate F such that G is an expansion of F. Moreover, F has the same kind as G.*

Proof. Let us discuss each of the conclusions of Theorem 38 separately.

If G contains a contractible bidiamond $\{u, v\}$, then let F be obtained from G by a contraction of this bidiamond, so that G is obtained from F by a decontraction of a bidiamond. Since the bidiamond is contractible, F is a simple graph and every triangle in F bounds a face, and thus F is a weak 4-candidate. Moreover, every 4-coloring φ of F extends to a 4-coloring of G: Let x and y be the common neighbors of u and v, let u' and v' be the neighbors of u and v, respectively, not in $\{u, v, x, y\}$, and let p be the vertex of F obtained by identifying x with y. Without loss of generality, we can assume that $\varphi(p) = 1$, $\varphi(u') = 2$, and $\varphi(v') \in \{2, 3\}$. We give x and y the color 1, u the color 3, and v the color 4. Therefore, since G is restrictive, F is restrictive as well, and has the same kind.

If G contains a 1-contractible path P, then let F be obtained from G/P by deleting all (at most two) triangle-isolated vertices, so that F is a weak 4-candidate. Observe that G is obtained from F by a decontraction of a 1-contractible path. Moreover, by Observation 31, the weak 4-candidate F is restrictive and has the same kind as G.

Finally, suppose that H is a proper amoeba in G, with vertices labeled as in the definition of an amoeba. We can choose H so that the eye of H is as large as possible; thus, the vertices y and v_3 do not have any common neighbor not belonging to the eye. Let F be obtained from G by deleting the internal vertices of the eye and adding the edge yv_3. Then F is a weak 4-candidate and G is obtained from F by amoeba expansion. Moreover, by Lemma 37, the weak 4-candidate F is restrictive (actually, even the canvas $F - yv_3$ is restrictive) and has the same kind as G. □

By induction, this has the following consequence.

Corollary 40. *If Conjecture 2 holds, then for every restrictive weak 4-candidate G, there exists a sequence $G_0, G_1, \ldots, G_m = G$ of restrictive*

weak 4-candidates of the same kind as G such that G_0 is trivial and for every $i \in \{1, \ldots, m\}$, the weak 4-candidate G_i is an expansion of G_{i-1}.

Note that the diamond is the only trivial weak 4-candidate of kind `d` and W_4 is the only trivial weak 4-candidate of kind `r`. Since the graphs in $\mathcal{C}_0$ are exactly the graphs obtained from weak 4-candidates by adding a crossing to the outer face, Corollary 40 implies Theorem 6, with the operation of expansion naturally lifted to the graphs in $\mathcal{C}_0$.

4 How to prove Conjecture 2?

It is easy to see that Conjecture 2 is a strengthening of the Four Color Theorem (indeed, it is well-known and straightforward to prove that a minimal counterexample to the Four Color Theorem would have minimum degree five and no separating triangles). Up to some minor variations, we know essentially only one proof of the Four Color Theorem: Using Kempe chains, prove that many configurations in planar graphs are *reducible*, i.e., cannot appear in a minimal counterexample, then use discharging to show that at least one of them appears in every planar graph. It is natural to ask whether a similar approach can be used to prove Conjecture 2. This indeed seems promising, but there are several issues to overcome (we state them in the setting of restrictive 4-candidates, which is equivalent by Corollary 12).

- Some of the Kempe chains may contain precolored vertices. This limits how we can exchange colors on Kempe chains, and results in fewer of the configurations being reducible.
- Reducibility arguments often involve replacing a configuration by a smaller one. This could decrease degrees of vertices or create non-facial triangles, and thus we would not obtain a smaller counterexample and reach a contradiction.
- The reducible configurations typically cannot contain precolored vertices, and this affects the discharging procedure.

The last issue is the least severe, since there is a standard technique for dealing with it by allocating extra charge for the precolored vertices. We discuss the first two issues in more detail in the rest of this section. Let us start with a useful observation.

Lemma 41. *Let G be a restrictive weak 4-candidate, let K be a 4-cycle in G, and let H be the subcanvas of G drawn in the closed disk bounded by K.*

If H is not restrictive, then the graph G_1 obtained from G by deleting the internal vertices of H is restrictive. Moreover, if $u, v \in V(K)$ are adjacent internal vertices whose degree in G_1 is four, then the graph $G_1 - \{u, v\}$ is also restrictive.

Proof. Let Q be the cycle bounding the outer face of G, and let ψ be a 4-coloring of Q which does not extend to a 4-coloring of G. Suppose for a contradiction that ψ extends to a 4-coloring φ_1 of G_1, and let ψ_1 be the restriction of φ_1 to K. Then ψ_1 does not extend to a 4-coloring of H, and since H is not restrictive, it follows that ψ_1 is bichromatic. Let $K = v_1v_2v_3v_4$; we can assume that $\psi_1(v_1) = \psi_1(v_3) = 1$ and $\psi_2(v_2) = \psi_2(v_4) = 2$.

For any distinct colors c_1 and c_2, let G_{c_1,c_2} be the subgraph of G_1 induced by the vertices to which φ_1 assigns colors c_1 and c_2. For a vertex v such that $\varphi_1(v) \in \{c_1, c_2\}$, let $G_{c_1,c_2}(v)$ be the component of G_{c_1,c_2} containing v. If $G_{1,3}(v_1)$ contains v_3, then observe that by planarity, there exists $i \in \{2, 4\}$ such that $G_{2,4}(v_i)$ is disjoint from Q and does not contain v_{6-i}. However, then we can exchange the colors 2 and 4 on $G_{2,4}(v_i)$, turning φ_1 into a 4-coloring φ_3 of G_1 extending ψ and whose restriction to K is v_1-diagonal. Since H is not restrictive, φ_2 further extends to a 4-coloring of G, which is a contradiction. If $G_{1,3}(v_1)$ does not contain v_3 and does not intersect Q, then we can similarly exchange colors 1 and 3 on $G_{1,3}(v_1)$ to turn φ_1 into a 4-coloring which is v_2-diagonal and extends ψ, then further extend it to a 4-coloring of G, again obtaining a contradiction.

We conclude that $G_{1,3}(v_1)$ does not contain v_3 and intersects Q. By symmetry, for every $i \in \{1, \ldots, 4\}$ and $c \in \{3, 4\}$, the subgraph $G_{\psi_1(v_i),c}(v_i)$ intersects K only in v_i and intersects Q. Note that this implies that for every $x \in V(G_{1,3}(v_1) \cap Q)$ and $y \in V(G_{1,3}(v_3) \cap Q)$, the vertices x and y are distinct and non-adjacent. Therefore, letting $Q = u_1u_2u_3u_4$, we can assume that $\psi(u_1), \psi(u_3) \in \{1, 3\}$ and $\psi(u_2), \psi(u_4) \in \{2, 4\}$.

If $u_1 \in V(G_{1,4}(v_1) \cap Q) \cup V(G_{1,4}(v_3) \cap Q)$, we similarly conclude that $\psi(u_1), \psi(u_3) \in \{1, 4\}$ and $\psi(u_2), \psi(u_4) \in \{2, 3\}$, and thus $\psi(u_1) = \psi(u_3) = 1$ and $\psi(u_2) = \psi(u_4) = 2$. Otherwise, we have $V(G_{1,4}(v_1) \cap Q) \cup V(G_{1,4}(v_3) \cap Q) = \{u_2, u_4\}$, $\psi(u_2), \psi(u_4) \in \{1, 4\}$ and $\psi(u_1), \psi(u_3) \in \{2, 3\}$, and thus $\psi(u_1) = \psi(u_3) = 3$ and $\psi(u_2) = \psi(u_4) = 4$. We conclude that ψ is bichromatic, which is a contradiction. Therefore, G_1 is restrictive.

Suppose now that say v_1 and v_2 are internal vertices whose degree in G_1 is four, and suppose for a contradiction that ψ extends to a 4-coloring φ_1' of $G_2 = G_1 - \{v_1, v_2\}$. Since G is a weak 4-candidate, G_1 contains triangles $v_1v_4w_1$, $v_2v_3w_2$, v_1v_2z, v_1w_1z, and v_2w_2z bounding faces, and $K' = w_1zw_2v_3v_4$ is a cycle bounding a face of G_2. Since G_1 is restrictive, φ_1'

cannot be extended to v_1 and v_2, and thus we can assume that $\varphi'_1(z) = 1$, $\varphi'_1(w_1) = \varphi'_1(v_3) = 2$, and $\varphi'_1(w_2) = \varphi'_1(v_4) = 3$.

For any distinct colors c_1 and c_2, let G'_{c_1,c_2} be the subgraph of G_2 induced by the vertices to which φ'_1 assigns colors c_1 and c_2. For a vertex v such that $\varphi'_1(v) \in \{c_1, c_2\}$, let $G'_{c_1,c_2}(v)$ be the component of G'_{c_1,c_2} containing v. If $G'_{1,3}(v_4)$ contains z, then observe that by planarity, there exists $u \in \{v_3, w_1\}$ such that $G'_{2,4}(u)$ is disjoint from Q and does not contain the vertex of $\{v_3, w_1\} \setminus \{u\}$. However, then we can exchange the colors 2 and 4 on $G'_{2,4}(u)$, turning φ'_1 into a 4-coloring of G_2 which extends ψ and which can be extended to a 4-coloring of G_1, which is a contradiction. Therefore, $z \notin V(G'_{1,3}(v_4))$ and $v_4 \notin V(G'_{1,3}(z))$. Let us remark that w_2 always belongs to $V(G'_{1,3}(z))$. If $G'_{1,3}(v_4)$ or $G'_{1,3}(z)$ is disjoint from Q, then we can similarly exchange the colors 1 and 3 on $G'_{1,3}(v_4)$ or $G'_{1,3}(z)$ and turn φ'_1 into a 4-coloring of G_2 which extends ψ and which can be extended to a 4-coloring of G_1, again obtaining a contradiction.

We conclude that $G'_{1,3}(v_4)$ and $G'_{1,3}(z)$ are disjoint and intersect Q. By planarity, $G'_{2,4}(v_3)$ and $G'_{2,4}(w_1)$ are disjoint, and a similar argument shows that they intersect Q. Symmetrically, $G'_{1,2}(v_3)$ and $G'_{1,2}(z)$ are disjoint and intersect Q, and $G'_{3,4}(v_4)$ and $G'_{3,4}(w_2)$ are disjoint and intersect Q. As in the previous case, this is only possible if ψ is bichromatic, which is a contradiction. □

In particular, using this lemma, we can exclude separating 4-cycles from a minimal counterexample to Conjecture 2.

Corollary 42. *Let G be a restrictive 4-candidate and let K be a 4-cycle in G that does not bound the outer face. If no 4-candidate with less that $|V(G)|$ vertices is restrictive, then the cycle K is hollow.*

Proof. Let Q be the 4-cycle bounding the outer face of G. Suppose for a contradiction that the subcanvas H of G drawn in the closed disk bounded by K has an internal vertex. Let G_1 be the subgraph of G obtained by deleting the internal vertices of H, and let f be the 4-face of G_1 bounded by K. We can assume that the 4-cycle K is chosen so that H is maximal, and thus no 4-cycle in G_1 other than Q and K bounds an open disk containing f. If K contains adjacent internal vertices u and v whose degree in G_1 is four, then let $G_2 = G_1 - \{u, v\}$, otherwise let $G_2 = G_1$. By Lemma 41, the graph G_2 is restrictive.

Next, iteratively repeat the following procedure: Let f' be the face of G_2 containing f. If G_2 contains an induced path P of length at most two with ends incident with f' and otherwise disjoint from the boundary of f',

then let Δ be the minimal closed disk containing f' and P, delete from G_2 the vertices and edges drawn in the interior of Δ, and repeat.

Let G_3 be the resulting graph. Using Observation 33, it is easy to see that every 4-coloring of G_3 extends to a 4-coloring of G_2, and thus G_3 is restrictive. Furthermore, observe that

- all faces of G_3 except for f' and the outer one are triangles,
- all internal vertices of G_3 not incident with f' have degree at least five,
- the face f' is bounded by an induced cycle K' and no vertex of $V(G_3) \setminus V(K')$ has non-adjacent neighbors in K', and in particular all internal vertices of K' have degree at least four, and
- either $G_3 = G_1$, or K' has length at least five.

If $G_3 \neq G_1$, then let G' be the graph obtained from G_3 by adding a vertex adjacent to all vertices of K' and drawing it inside f'. Then G' is a restrictive 4-candidate with fewer vertices than G, which is a contradiction.

Hence, suppose that $G_3 = G_1$. The construction of G_2 implies that no two internal vertices of K of degree four are adjacent. Hence, there exist non-adjacent vertices $x, y \in V(K)$ such that neither of the two vertices of $V(K) \setminus \{x, y\}$ is an internal vertex of degree four. In this case, we let $G' = G_3 + xy$. Again, observe that G' is a restrictive 4-candidate with fewer vertices than G, which is a contradiction. □

This allows us to exclude reducible configurations which do not need to be replaced by smaller ones. More precisely, the following claim holds.

Corollary 43. *Let G be a restrictive 4-candidate and let K be a cycle in G such that the open disk Λ bounded by K contains at least two vertices. Let G_1 be the graph obtained from G by deleting the vertices and edges drawn in Λ. If no 4-candidate with less that $|V(G)|$ vertices is restrictive, then G_1 is not restrictive.*

Proof. Suppose for a contradiction that G_1 is restrictive. Iteratively repeat the following procedure: Let f be the face of G_1 containing Λ. If G_1 contains an induced path P of length at most two with ends incident with f and otherwise disjoint from the boundary of f, then let Δ be the minimal closed disk containing f and P, delete from G_1 the vertices and edges drawn in the interior of Δ, and repeat.

Let G_2 be the resulting graph. By Observation 33, every 4-coloring of G_2 extends to a 4-coloring of G_1, and thus G_2 is restrictive. Moreover, all

faces of G_2 except for f and the outer one are triangles, all internal vertices of G_2 not incident with f have degree at least five, the face f is bounded by an induced cycle K' and no vertex of $V(G_2) \setminus V(K')$ has non-adjacent neighbors in K', and in particular all internal vertices of K' have degree at least four. Moreover, Corollary 42 implies that K' has length at least five.

Let G' be obtained from G_2 by adding a vertex adjacent to all vertices of K' and drawing it inside f. Then G' is a restrictive 4-candidate with fewer vertices than G, which is a contradiction. □

Let θ be a (diagonal or rainbow) 4-coloring of the cycle Q bounding the outer face of G which does not extend to a 4-coloring of G, and let H be the subcanvas of G drawn in the closure of Λ. Corollary 43 implies that if we can show (using Kempe chains and the properties of H) that every 4-coloring of G_1 extending θ can be turned into a 4-coloring of G with the same restriction to Q, then we obtain a contradiction. Since the procedure for obtaining such proofs (showing that H is *D-reducible*) is standard (see e.g. [11] for a detailed description) and requires only minor modifications to fit our setting, we leave its presentation to Appendix.

As shown in [12], D-reducibility arguments are sufficient to prove the Four Color Theorem. However, many of the configurations that are D-reducible in the setting of the Four Color Theorem are not D-reducible in our setting, since we get extra constraints on switching Kempe chains containing the precolored vertices (in fact, small D-reducible configurations seem to be very rare—there is some hope that for large configurations, the loss of a few Kempe chains because of the precolored vertices becomes less important).

Realistically, to prove the conjecture, we will need to use *C-reducible* configurations, where the configuration H is replaced by a smaller configuration H' (thus, we only need to argue that every 4-coloring of the resulting graph G' which extends θ can be turned into a 4-coloring of G with the same restriction to Q); note that this replacement can also involve identification of the vertices of the cycle K bounding the configuration. An issue that proofs of the Four Color Theorem using C-reducible configurations need to overcome is that such a replacement could result in creation of loops, making the reduced graph impossible to color (this is dealt with by observing that the loops correspond to short separating cycles in the original graph, which are excluded using arguments similar to the proof of Corollary 42). In our setting, the issue is compounded by the possibility that G' is not a candidate, i.e., that the replacement of H by H' can create internal vertices of degree at most four or non-facial triangles (and no reduction along the lines

of Corollary 43 may be possible). One way to deal with the issue would be to find an exact (or at least sufficiently detailed) characterization of restrictive weak 4-candidates based on the assumption that the conjecture holds for graphs with less than $|V(G)|$ vertices, and then use it to argue that G' cannot be restrictive even if it is not a candidate.

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Appendix

In this Appendix, we give a precise description of the standard Kempe chain reducibility argument in our setting.

A *color partition* is an equivalence π on the set $\{1, \ldots, 4\}$ which has exactly two equivalence classes of size two. Let Q and K be disjoint cycles drawn in the plane so that K is contained in the closed disk bounded by Q, and let ψ be a 4-coloring of $Q \cup K$. Let ψ/π be the equivalence on $V(Q \cup K)$ such that $u \equiv_{\psi/\pi} v$ if and only if $\psi(u) \equiv_\pi \psi(v)$. An edge uv of $Q \cup K$ is *ψ/π-monochromatic* if $u \equiv_{\psi/\pi} v$. A *potential π-Kempe chain extract* for ψ is a triple (β, κ, σ) of equivalences on $V(Q \cup K)$ such that

(i) $\beta = \psi/\pi$,

(ii) κ is a refinement of ψ/π such that $u \equiv_\kappa v$ for every ψ/π-monochromatic edge $uv \in E(Q \cup K)$,

(iii) σ is the refinement of κ such that for every $u, v \in V(Q \cup K)$, we have $u \equiv_\sigma v$ if and only if $u \equiv_\kappa v$ and $\psi(u) = \psi(v)$, and

(iv) there exists a plane graph F consisting of a perfect matching M on the non-ψ/π-monochromatic edges of $E(Q \cup K)$ and possibly of a cycle C vertex-disjoint from the matching, such that

   - the drawing of F is contained in the annulus Θ between K and Q and intersects the boundary of Θ only in the points representing the vertices of M, equal to the midpoints of the drawings of the corresponding edges of $Q \cup K$, and
   - any two vertices of $V(Q \cup K)$ are κ-equivalent if and only if they are drawn in the same arcwise-connected component of the space obtained from Θ by removing the drawing of F.

We say that such a plane graph F is a *realizer* for the potential π-Kempe chain extract. Let us remark that whether such a realizer exists can of course be determined from the combinatorial properties of κ. Informally, the triple (β, κ, σ) describes a possibility how Kempe chains of a 4-colored plane graph with facial cycles K and Q could look like:

- We simultaneously consider Kempe chains for two disjoint pairs $A = \{c_1, c_2\}$ and $B = \{c_3, c_4\}$ of colors, and β divides $V(Q \cup K)$ into vertices whose color belongs to A and vertices whose colors belong to B (without explicitly saying which equivalence class corresponds to A

and which to B—this does not matter, since the two possibilities are equivalent up to a permutation of colors).

- The classes of the equivalence κ correspond to the intersections of the Kempe chains with $V(Q \cup K)$; the condition (iv) expresses the planarity constraints on Kempe chains, as we discuss below.

- The equivalence σ divides each class of κ into (at most) two parts based on the colors of vertices (without explicitly specifying the colors corresponding to the equivalence classes, since the colors can be exchanged on the Kempe chain).

An *environment* is a triple (G, Q, K), where G is a plane graph such that the cycle $Q \subseteq G$ bounds the outer face of G, the cycle $K \subseteq G$ bounds an internal face of G, and all other faces of G have length three. Let φ be a 4-coloring of G. We are going to consider the Kempe chains of this 4-coloring for two disjoint pairs $\{c_1, c_2\}$ and $\{c_3, c_4\}$ of colors. Let π be the color partition such that $c_1 \equiv_\pi c_2$ and $c_3 \equiv_\pi c_4$, let ψ be the restriction of φ to $Q \cup K$, and let $\beta = \psi/\pi$. Let $X \subseteq E(G)$ consist of the edges uv such that $\varphi(u) \not\equiv_\pi \varphi(v)$, so that $G - X$ is the disjoint union of the $\{c_1, c_2\}$ and $\{c_3, c_4\}$ Kempe chains of φ, and let κ be the equivalence on $V(Q \cup K)$ where two vertices are equivalent if and only if they belong to the same component of $G - X$. Let σ be the equivalence on $V(Q \cup K)$ such that two vertices are equivalent if and only if they are κ-equivalent and have the same color.

We claim that (β, κ, σ) is a potential π-Kempe chain extract for ψ. Indeed, the conditions (i), (ii), and (iii) are clearly satisfied, and thus it suffices to find a realizer showing that (iv) holds. Let F_1 be the spanning subgraph of the dual of G containing exactly the duals of the edges of X, and let k and q be the vertices of F_1 corresponding to the faces of G bounded by K and Q, respectively. Observe that all vertices of F_1 have even degree; in particular, all vertices other than k and q have degree 0 or 2. Hence, the drawing of F_1 can be decomposed into simple curves from k to q and simple closed curves passing through at most one of k and q, disjoint everywhere except for k and q. Moreover, note that that each $\{c_1, c_2\}$ and $\{c_3, c_4\}$ Kempe chain consists exactly of the vertices of G drawn in a face of F_1. Thus, F_1 can be turned into a realizer of (β, κ, σ) by

- deleting all components not containing k and q, except for one cycle separating k from q if any is present,

- subdividing each edge $e^\star$ of F_1 dual to an edge e of $Q \cup K$ by a vertex drawn at the intersection of these edges, and

- deleting the vertices k and q and suppressing the vertices of degree two.

We say that (β, κ, σ) is the *π-Kempe chain extract* of φ.

For a potential π-Kempe chain extract (β, κ, σ) for a 4-coloring ψ of $Q \cup K$, let $n_{\psi,\pi,\beta,\kappa,\sigma}(G)$ denote the number of 4-colorings of G which extend ψ and whose π-Kempe chain extract is equal to (β, κ, σ). Let us note the following key observation.

Observation 44. *Let (G, Q, K) be an environment, and for $i \in \{1, 2\}$, let π_i be a color partition and ψ_i a 4-coloring of $Q \cup K$. Suppose that (β, κ, σ) is both a potential π_1-Kempe chain extract for ψ_1 and a potential π_2-Kempe chain extract for ψ_2. Then*

$$n_{\psi_1,\pi_1,\beta,\kappa,\sigma}(G) = n_{\psi_2,\pi_2,\beta,\kappa,\sigma}(G).$$

Proof. By permuting the colors if needed, we can assume that π_1 and π_2 are both equal to the color partition π such that $1 \equiv_\pi 2$ and $3 \equiv_\pi 4$. Since $\psi_1/\pi = \psi_2/\pi = \beta$, by exchanging the colors 1 and 2 with colors 3 and 4 in ψ_2 if needed, we can furthermore assume that $\psi_1(v) \equiv_\pi \psi_2(v)$ for every $v \in V(Q \cup K)$. Because ψ_1 and ψ_2 also share the part σ of the potential π-Kempe chain extract, this implies that there exists a set S of equivalence classes of κ such that ψ_2 is obtained from ψ_1 by exchanging the color 1 with 2 and the color 3 with 4 on the vertices of the elements of S.

Let φ be any 4-coloring of G extending ψ_1 whose π-Kempe chain extract is (β, κ, σ). For distinct colors c_1 and c_2, let G_{c_1,c_2} be the subgraph of G induced by the vertices of colors c_1 and c_2. The definition of the π-Kempe chain extract implies that for each equivalence class C of κ, there exists a component G_C of $G_{1,2} \cup G_{3,4}$ intersecting $V(Q \cup K)$ exactly in C. By exchanging the color 1 with 2 and the color 3 with 4 on $\bigcup_{C \in S} G_C$, we transform φ into a 4-coloring of G extending ψ_2 whose π-Kempe chain extract is (β, κ, σ). It is easy to see that this correspondence establishes a bijection proving the equality

$$n_{\psi_1,\pi_1,\beta,\kappa,\sigma}(G) = n_{\psi_2,\pi_2,\beta,\kappa,\sigma}(G).$$

□

Thus, let us define $n_{\beta,\kappa,\sigma}(G)$ as $n_{\psi,\pi,\beta,\kappa,\sigma}(G)$ for any 4-coloring ψ and color partition π such that (β, κ, σ) is a potential π-Kempe chain extract for ψ. Let us also define Ψ as the set of all 4-colorings of $K \cup Q$ and $\mathcal{E}$ as the

set of all potential Kempe chain extracts, and for every $\psi \in \Psi$, let $n_\psi(G)$ be the number of 4-colorings of G that extend ψ.

Consider now an environment (G, Q, K) and a plane graph F with the outer face bounded by K. Suppose that θ is a 4-coloring of Q that does not extend to 4-coloring of $G \cup F$. Thus, $n_\psi(G) = 0$ for every 4-coloring ψ of $Q \cup K$ that extends θ. Together with Observation 44 and the obvious fact that the number of 4-colorings subject to any condition is non-negative, this gives a set of linear constraints satisfied by the vector $(n_i(G) : i \in \Psi \cup \mathcal{E})$.

More precisely, let Π be the set of all three color partitions, and for $\psi \in \Psi$ and $\pi \in \Pi$, let $E_{\psi,\pi} \subseteq \mathcal{E}$ denote the set of all potential π-Kempe chain extracts for ψ. For a 4-coloring θ of Q, let $\Psi_{F,\theta}$ be the set of 4-colorings $\psi \in \Psi$ such that the restriction of ψ to Q is θ and such that ψ extends to a 4-coloring of F. We define $\Phi_{F,\theta}$ as the cone (independent of the environment G) consisting of the points $x \in \mathbb{R}^{\Psi \cup \mathcal{E}}$ satisfying the following constraints:

$$\begin{aligned} x &\geq 0 \\ x(\psi) &= \sum_{\epsilon \in E_{\psi,\pi}} x(\epsilon) && \text{for every } \psi \in \Psi \text{ and } \pi \in \Pi \\ x(\psi) &= 0 && \text{for every } \psi \in \Psi_{F,\theta}. \end{aligned}$$

A coloring ω of K is (F, θ)-*feasible* if there exists $x \in \Phi_{F,\theta}$ such that $x(\omega \cup \theta) \neq 0$; that is, the set of constraints defining $\Phi_{F,\theta}$ does not exclude the possibility that G has a 4-coloring extending θ whose restriction to K equals ω. Otherwise, we say that ω is (F, θ)-*infeasible*. In particular, every 4-coloring of K which extends to a 4-coloring of F is (F, θ)-infeasible.

Let F_1 be a plane graph with the outer face bounded by a closed walk W of the same length as K and let f be a function mapping the vertices of K to vertices of W in order. Note that for a 4-coloring ω_1 of W, $f \circ \omega_1$ is a 4-coloring of K. We say that (F_1, f) is a θ-*reducent* for F if for every 4-coloring ω_1 of W which extends to a 4-coloring of F_1, the 4-coloring $f \circ \omega_1$ is (F, θ)-infeasible. Let G_1 be the graph obtained from G by adding F_1 and identifying each vertex $v \in V(K)$ with the vertex $f(v)$; we say that G_1 is obtained from $G \cup F$ by *replacing* F by (F_1, f). The basic property used to show reducibility of configurations is given in the following observation.

Observation 45. *Let G_0 be a canvas with the outer face bounded by a cycle Q and let K be a cycle in G disjoint from Q. Let F be the subgraph of G_0 drawn in the closed disk bounded by K and let G be the subgraph of G_0 drawn in the closed annulus between K and Q, so that (G, Q, K) is an environment. Let θ be a 4-coloring of Q and let (F_1, f) be a θ-reducent for*

F. Let G_1 be the graph obtained from G_0 by replacing F by (F_1, f). If θ does not extend to a 4-coloring of G_0, it also does not extend to a 4-coloring of G_1.

Proof. Since θ does not extend to a 4-coloring of G_0, we have $n_\psi(G) = 0$ for every $\psi \in \Psi_{F,\theta}$. Moreover, for every $\psi \in \Psi$ and $\pi \in \Pi$, every 4-coloring of G which extends ψ has a uniquely determined π-Kempe chain extract, and thus

$$n_\psi(G) = \sum_{\epsilon \in E_{\psi,\pi}} n_{\psi,\pi,\epsilon}(G) = \sum_{\epsilon \in E_{\psi,\pi}} n_\epsilon(G).$$

Therefore, the point $x \in \mathbb{R}^{\Psi \cup \mathcal{E}}$ such that $x(\psi) = n_\psi(G)$ for $\psi \in \Psi$ and $x(\epsilon) = n_\epsilon(G)$ for $\epsilon \in \mathcal{E}$ belongs to $\Phi_{F,\theta}$.

Suppose for a contradiction that φ_1 is a 4-coloring of G_1 extending θ, and let ω_1 be the restriction of φ_1 to W. Clearly ω_1 extends to a 4-coloring of F_1, and since (F_1, f) is a θ-reducent, the 4-coloring $f \circ \omega_1$ of K is (F, θ)-infeasible.

Let φ be the 4-coloring of G corresponding to φ_1; that is, $\varphi(v) = \varphi_1(v)$ for $v \in V(G) \setminus V(K)$ and $\varphi(v) = \varphi_1(f(v))$ for $v \in V(K)$. Let ψ be the restriction of φ to $Q \cup K$. Since ψ extends to a 4-coloring φ of G, we have

$$0 < n_\psi(G) = x(\psi).$$

Observe that $\psi = \theta \cup (f \circ \omega_1)$, and thus the 4-coloring $f \circ \omega_1$ is (F, θ)-feasible. This is a contradiction. □

In the situation of Observation 45, if $|V(F)| > |V(K)|$ and $F_1 = K$ (i.e., all 4-colorings of K are (F, θ)-infeasible), then we say that F is *D-reducible*. In case that we know that θ extends to a 4-coloring of every non-trivial environment in G_0 (e.g., as in Corollary 43), this implies that the configuration F cannot appear in G_0. If F_1 is any graph with $|V(F_1)| < |V(F)|$, then F is *C-reducible with reducent* F_1. As we discuss at the end of Section 4, there are extra challenges in being able to use C-reducible configurations, since it is harder to ensure that induction hypothesis can be applied to G_1.

Finally, let us remark that in the presented formulation (called block-count reducibility [4, 8]), determining whether a coloring is (F, θ)-feasible involves solving a linear program; this is generally too slow to be practical for large configurations. Instead, one usually solves the boolean restriction of the program, with variables $x(\psi)$ and $x(\epsilon)$ being true when $n_\psi(G) > 0$ and $n_\epsilon(G) > 0$, respectively, and with addition replaced by logical disjunction. In this boolean formulation, the program can be solved by straightforward greedy propagation, see e.g. [11] for more details.