Irreversible 2-conversion set in graphs of bounded degree*

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Abstract

An *irreversible k-threshold process* (also a *k-neighbor bootstrap percolation*) is a dynamic process on a graph where vertices change color from white to black if they have at least k black neighbors. An *irreversible k-conversion set* of a graph G is a subset S of vertices of G such that the irreversible k-threshold process starting with S black eventually changes all vertices of G to black. We show that deciding the existence of an irreversible 2-conversion set of a given size is NP-complete, even for graphs of maximum degree 4, which answers a question of Dreyer and Roberts. Conversely, we show that for graphs of maximum degree 3, the minimum size of an irreversible 2-conversion set can be computed in polynomial time. Moreover, we find an optimal irreversible 3-conversion set for the toroidal grid, simplifying constructions of Pike and Zou.

Keywords: irreversible k-conversion process; spread of infection; bootstrap percolation; NP-complete problem; matroid parity problem; toroidal grid

1 Introduction

Mathematical modelling of the spread of infectious diseases was recently studied by Roberts [23] and by Dreyer and Roberts [13]. They used the following model.

Let $G = (V, E)$ be a graph with vertices colored white and black. An *irreversible k-threshold process* is a process where vertices change color from white to black. More precisely, a white vertex becomes black at time $t + 1$ if at least k of its neighbors are black at time t.

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An *irreversible k-conversion set* S is a subset of V such that the irreversible k-threshold process starting with vertices of S set to black and all other white will result in a graph G with all vertices black after finite number of steps.

More general models of spread of infectious diseases and the complexity of the related problems were studied by Boros and Gurwich [9].

A natural question to ask is what is the minimum size of an irreversible k-conversion set in a graph G.

Problem IkCS(G, s):

Input:	a graph G and a positive integer s	
Output:	YES	if there exists an irreversible k-conversion set of size s in G
	NO	otherwise

Dreyer and Roberts [13] proved that IkCS is NP-complete for every fixed $k \geq 3$ by an easy reduction from the independent set problem. For $k = 1$ the problem is trivially polynomial since one black vertex per connected component is necessary and sufficient. Dreyer and Roberts [13] asked what is the complexity of the IkCS problem if $k = 2$. As the first result of this paper we resolve this open question.

Theorem 1. *The problem I2CS is NP-complete even for graphs of maximum degree* 4*.*

A subset W of vertices of a graph $G = (V, E)$ is a *vertex feedback set* if $V \setminus W$ is acyclic. For 3-regular graphs, the I2CS problem is equivalent to finding a vertex feedback set, which can be solved in polynomial time [25]. We extend this result to graphs of maximum degree 3.

Theorem 2. *The problem I2CS is polynomially solvable for graphs of maximum degree* 3*.*

Boros and Gurwich [9] proved that if every vertex has its own threshold, then determining the minimum size of the conversion set in graphs of maximum degree 3 is NP-complete. Note that the problem I2CS(G, s) is trivially polynomial if the maximum degree of G is at most 2 as a path of length l requires $\lceil \frac{l+1}{2} \rceil$ black vertices and a cycle of length l requires $\lceil \frac{l}{2} \rceil$ vertices.

We also give a construction of an optimal irreversible 3-conversion set for a toroidal grid $T(m, n)$, which is the Cartesian product of the cycles C_m and C_n. Flocchini et al. [14] and Luccio [20] gave lower and upper bounds differing by a linear $O(m + n)$ term; see also [13]. Pike and Zou [22] gave an optimal construction. We present a simpler optimal construction.

Theorem 3. *Let T be a toroidal grid of size $m \times n$, where $m, n \geq 3$. If $n = 4$ or $m = 4$ then T has an irreversible 3-conversion set of size at most $\frac{3mn+4}{8}$. Otherwise, T has an irreversible 3-conversion set of size at most $\frac{mn+4}{3}$.*

Theorems 1 and 3 appeared in our early preprint [17]. When preparing the final version of this paper, we found that Centeno et al. [10] have published a different proof that the problem I2CS is NP-complete. The graph in their construction has maximum degree 11.

Balogh and Pete [6] reported tight asymptotic bounds on the minimum size of an irreversible k-conversion set in the d-dimensional integer grid. Balister, Bollobás, Johnson and Walters [3] obtained more precise bounds for the case $d + 1 \leq k \leq 2d$.

An irreversible k-conversion process is equivalently also called a *(k-neighbor) bootstrap percolation*. Bootstrap percolation was introduced by Chalupa, Leath and Reich [11] as a model for interactions in magnetic materials. Bootstrap percolation theory is typically

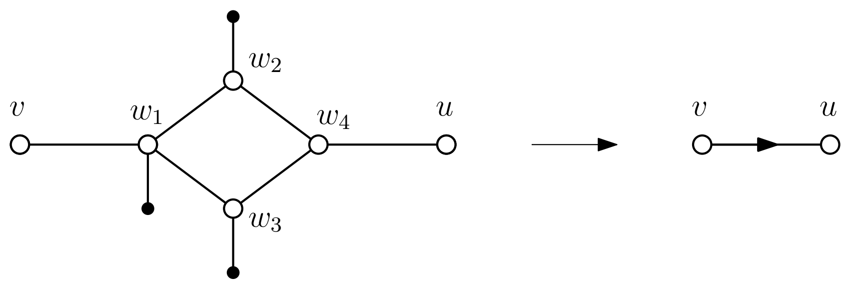


Figure 1: A one-way gadget.

concerned with d-dimensional lattices (and in recent years, other classes of graphs as well) where each vertex is "infected" independently at random with some fixed probability. See [1] for an early review of the subject or [12] for a recent survey. See [4, 26] for the most recent results for d-dimensional integer grids.

Several authors [7, 21] studied the computational complexity of the minimum number of steps of the bootstrap percolation needed to percolate the whole graph.

Many other variants of bootstrap percolation have been studied in the literature. Examples include hypergraph bootstrap percolation [5] or weak H-saturation of graphs [2, 8].

2 Irreversible 2-conversion set is NP-complete

In this section we give a proof of Theorem 1.

The problem is trivially in NP. A verification that $S \subseteq V$ is an irreversible 2-conversion set can be done by iterating the threshold process. It is enough to check only the first $|V|$ steps. Hence the verification can be done in a polynomial time.

In the rest of the proof we show that I2CS(G, s) is NP-hard by a polynomial-time reduction from 3-SAT. We introduce a variable gadget, a clause gadget and a gadget that checks if all clause gadgets are satisfied.

Since a white vertex needs two black neighbors to become black, we have the following observation.

Observation 4. *Every irreversible 2-conversion set contains all vertices of degree* 1.

According to this observation, in the figures of the gadgets we draw vertices of degree one black.

Let $\mathcal{F}$ be an instance of 3-SAT. We denote the number of variables by n and the number of clauses by m. We construct a graph $G_\mathcal{F}$ and give a number s such that $\mathcal{F}$ is satisfiable if and only if $G_\mathcal{F}$ has an irreversible 2-conversion set of size s.

First we introduce a *one-way* gadget; see Figure 1. The gadget contains two vertices u and v which are called *start* and *end* of the one-way gadget. Vertices w_1, w_2, w_3 and w_4 are called *internal* vertices of the gadget.

Observation 5. *Let u and v be start and end of the one-way gadget. If internal vertices are white at the beginning then the following holds:*

1. If v is black then u gets a black neighbor from the gadget in three steps.

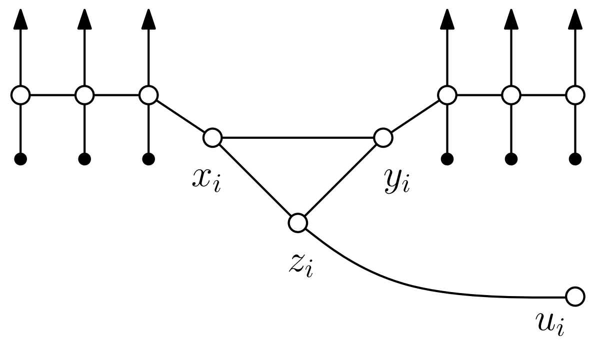


Figure 2: A variable gadget $g(X_i)$ connected to a vertex u_i of a distributing path.

2. *The vertex w_4 can become black only after v becomes black.*

We refer to the one-way gadget by a directed edge in the following figures. Later, we set s so that S cannot contain any internal vertices of one-ways. Thus, in the rest of the proof we assume that all internal vertices are white at the beginning.

2.1 Variable gadget

A gadget $g(X_i)$ for a variable X_i, where $1 \leq i \leq n$, consists of a triangle $x_i y_i z_i$ and two *antennas*; see Figure 2. The length of the antenna connected to x_i and y_i is equal to the number of occurrences of X_i and $\neg X_i$, respectively, in the clauses of $\mathcal{F}$. We call the white vertices of the x_i antenna *positive outputs* and the vertices of the y_i antenna *negative outputs*. One-way gadgets with starts in the outputs have ends in clause gadgets. The vertex z_i is adjacent to a vertex u_i lying on a distributing path, which we define later.

We show that exactly one of x_i and y_i is black at the beginning. This represents the value of the variable X_i. The vertex x_i corresponds to the true and y_i to the false evaluation of X_i. The purpose of the connection between u_i and z_i is to convert all vertices of the gadget to black if $\mathcal{F}$ is satisfiable.

Observation 6. *Let S be an irreversible 2-conversion set. The gadget $g(X_i)$ has the following properties.*

(a) If x_i is black then all positive outputs will become black in the process. Similarly for y_i and negative outputs.

(b) If two of x_i, y_i, z_i are black then all vertices of the gadget will become black in the process.

(c) S must contain at least one of the vertices x_i, y_i, z_i.

(d) If S contains exactly one vertex of the gadget (except the vertices of degree 1) then it must be x_i or y_i.

(e) If S contains exactly one vertex of the gadget then z_i gets black only if u_i gets black.

Proof. The first two properties are easy to check and hence we skip them.

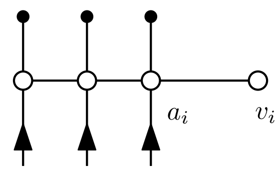


Figure 3: A clause gadget $g(C_i)$ connected to a vertex v_i of a collecting path.

First we check the property (c). Every vertex of the triangle $x_iy_iz_i$ has only one neighbor outside the triangle. Hence if all three vertices are white, they remain white forever since each of them has at most one black neighbor. Therefore S must contain at least one of them.

Now we check the property (d). If S is allowed to contain only one of $\{x_i, y_i, z_i\}$ then all positive and negative outputs are white at the beginning. Moreover, the positive outputs may become black only if x_i gets black. Similarly for negative outputs and y_i.

Suppose for contradiction that $z_i \in S$. Then both x_i and y_i have only one black neighbor (z_i) at the beginning. During the process the other black neighbor has to be some output vertex which is not possible. Hence S must contain x_i or y_i.

Finally, we check the property (e). By (d) we know that z_i is white at the beginning. Assume without loss of generality that y_i is also white while x_i is black. The vertex z_i can get black if y_i or u_i gets black. So assume for contradiction that y_i gets black before z_i. The only possibility is that the vertex from the antenna adjacent to y_i gets black. But it is not possible since output vertices are white at the beginning and they are connected to the rest of the graph by one-ways. □

Note that if x_i or y_i is in S then there is still a chance that the process converts all vertices of the gadget to black, as the vertex u_i may become black during the process.

Let L be a set of all degree 1 vertices in $G_{\mathcal{F}}$. We set the parameter s to $|L| + n$. Thus every variable gadget has exactly one of x_i and y_i black at the beginning and all other vertices of $G_{\mathcal{F}}$ of degree at least 2 are white. We compute $|L|$ after we describe all the remaining gadgets.

2.2 Clause gadget

The gadget $g(C_i)$ for a clause $C_i = (L_o \vee L_p \vee L_q)$, where $1 \leq i \leq m$ and L_o, L_p, and L_q are literals, is depicted in Figure 3. The gadget consists of a path on three vertices corresponding to the three literals in the clause. We call the path the *spine* of the clause gadget. Each vertex of the spine has one neighbor of degree 1 and is connected to the gadget of the corresponding variable by a one-way. The vertex of a clause corresponding to a literal X_i is connected to a positive output of $g(X_i)$ and the vertex corresponding to a literal $\neg X_i$ is connected to a negative output of $g(X_i)$. Finally, one vertex of the spine denoted by a_i is connected to a vertex v_i of a collecting path, which is defined later.

Observation 7. *If one vertex of the spine is black, then all vertices of the clause gadget get black in the process.*

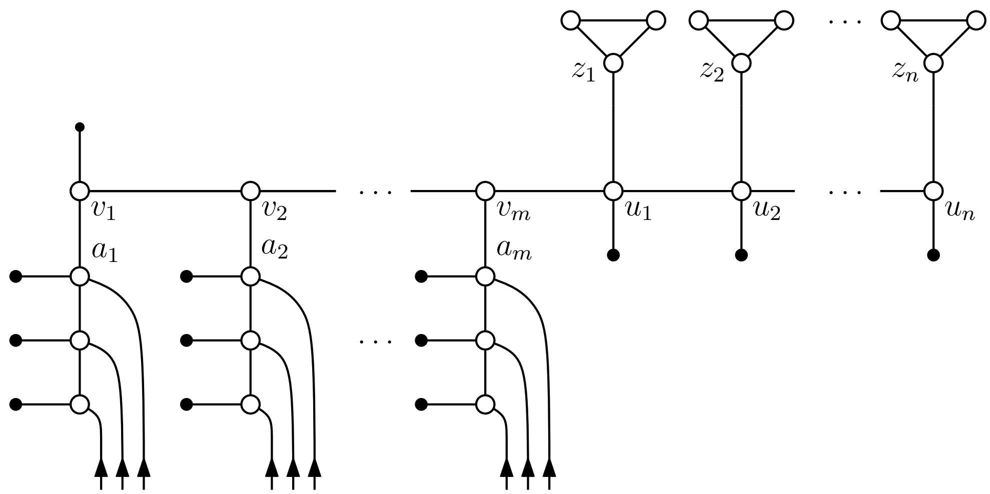


Figure 4: A collecting path $v_1, \dots, v_m$ and a distributing path $u_1, \dots, u_n$

2.3 Collecting and distributing gadget

A *collecting path* is a path on m vertices $v_1, \dots, v_m$ where each v_i is connected to a clause gadget. Moreover, the vertex v_1 is also connected to a vertex of degree 1. A *distributing path* is a path on n vertices $u_1, \dots, u_n$. Each u_i is connected to a vertex of degree 1 and to the vertex z_i of the variable gadget $g(X_i)$. Finally, v_m is connected to u_1; see Figure 4. See Figure 5 for an example of the whole graph $G_{\mathcal{F}}$.

Observation 8. *If the vertices of the distributing and collecting paths are white at the beginning they will become all black in the process only if all the clause gadgets get black during the process.*

Proof. If all spines of clause gadgets are black then it is easy to observe that the vertices of the collecting path get black in at most m steps from v_1 to v_m. Once v_m is black all the vertices of the distributing path get black in at most n steps from u_1 to u_n. It remains to check that v_i cannot get black before a neighboring vertex a_i gets black.

We start by checking the vertices of the distributing path. By Observation 6(e), the vertex z_n cannot get black before u_n. Thus u_n cannot get black before u_{n-1} because u_{n-1} is one of the two remaining neighbors which can be black before u_n. Similarly, for $0 < i < n$, the vertices z_i and u_{i+1} cannot get black before u_i. Thus u_i cannot get black before u_{i-1}. Similarly, u_0 cannot get black before v_m.

Analogously, no vertex v_i, $2 \leq i \leq m$, of the collecting path can get black before v_{i-1} and a_i are both black. For $i = 1$ we get that a_1 must get black before v_1. □

The graph $G_{\mathcal{F}} = (V, E)$ corresponding to the 3-SAT instance $\mathcal{F}$ constructed from these gadgets has a linear size in the size of $\mathcal{F}$. The size of L is $15m + n + 1$. Thus s is set to $n + |L| = 15m + 2n + 1$.

Lemma 9. *If $\mathcal{F}$ is satisfiable then there exists an irreversible 2-conversion set S of size $n+|L|$ in $G_{\mathcal{F}}$.*

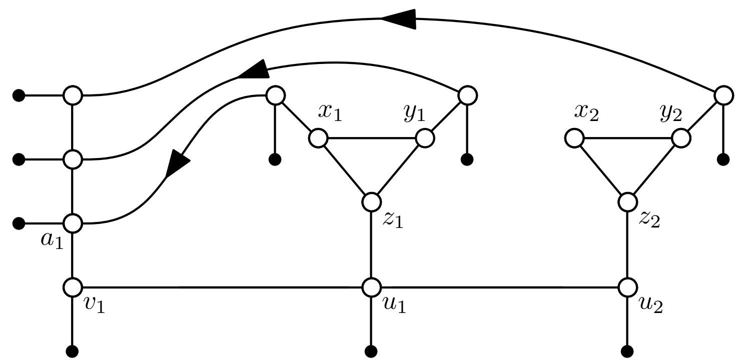


Figure 5: A graph $G_{\mathcal{F}}$ for the formula $\mathcal{F} = (X_1 \vee \neg X_1 \vee \neg X_2)$.

Proof. The set S consists of $|L|$ leaves and from every variable gadget $g(X_i)$ we choose either x_i or y_i if X_i is evaluated true or false, respectively. Since $\mathcal{F}$ is satisfiable then after a finite number of steps every gadget for a clause has at least one black vertex. Then in at most two steps all clause gadgets are completely black. Next the collecting path gets black in at most m steps and the distributing path gets black in next n steps. Now, for $0 \leq i \leq n$, the vertex z_i has two black neighbors and it gets black. The remaining white vertex of the pair x_i, y_i gets black in the next step. Finally, also the remaining antennas for every variable get black. Hence all vertices of $G_{\mathcal{F}}$ get black in the process. □

Lemma 10. *If $\mathcal{F}$ is not satisfiable then there is no irreversible 2-conversion set of size $n+|L|$.*

Proof. Assume for contradiction that there exists an irreversible 2-conversion set S of size $n + |L|$. By Observation 4, $L \subseteq S$. Moreover, due to Observation 6, S must contain one of $\{x_i, y_i\}$ for each $i \in [n]$. Hence there are no other black vertices. We derive the truth assignment of the variables in the following way. We set X_i true if $x_i \in S$ and false if $y_i \in S$.

Let $C = (L_o \vee L_p \vee L_q)$ be a clause of $\mathcal{F}$. The gadget corresponding to C gets black after a finite number of steps of the process. By Observation 8, $g(C)$ got black because of one of $g(X_o)$, $g(X_p)$ or $g(X_q)$. Hence C is evaluated as true in $\mathcal{F}$. Therefore all clauses of $\mathcal{F}$ are evaluated as true which is a contradiction with the assumption that $\mathcal{F}$ is not satisfiable. □

The proof of Theorem 1 is now finished.

3 Graphs with maximum degree 3

In this section we give a proof of Theorem 2. We follow the approach of Ueno, Kajitani and Gotoh [25].

Let G be a graph with maximum degree 3. Without loss of generality, we assume that G is connected, since a minimum irreversible 2-conversion set can be computed for each component separately. First we reduce the problem to graphs with minimum degree 2. Let H_5 be the graph with five vertices and seven edges consisting of the cycle $v_1v_2v_3v_4v_5$ and the edges v_2v_4 and v_3v_5. Let G_2 be the graph obtained from G by attaching a copy of H_5 to each vertex v of G of degree 1 by identifying v with v_1; see Figure 6. Observe that G_2 is a graph with maximum degree 3 and minimum degree at least 2.

For any graph H, let $C_2(H)$ be the minimum size of an irreversible 2-conversion set in H.

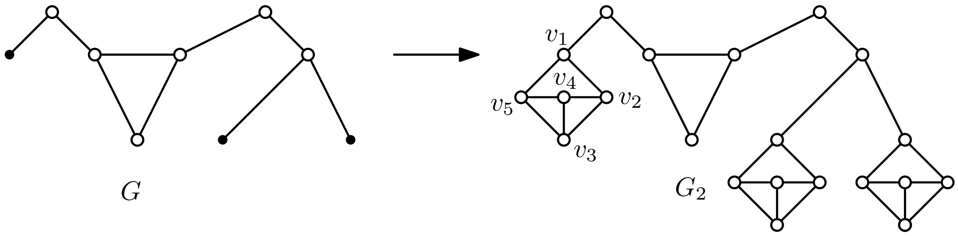


Figure 6: A reduction of the I2CS problem to graphs with all degrees at least 2.

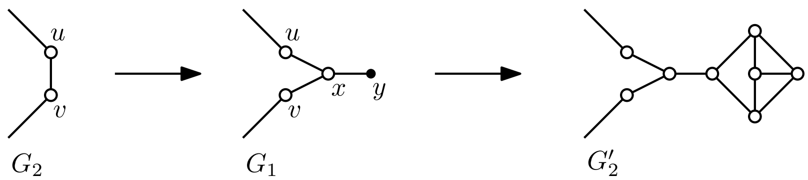


Figure 7: The case of two adjacent vertices of degree 2.

Lemma 11. *Let k be the number of vertices of degree 1 in G. Then $C_2(G_2) = C_2(G) + k$.*

Proof. By Observation 4, every irreversible 2-conversion set in G contains all vertices of degree 1. Since every irreversible 2-conversion set in G_2 intersects all cycles, it contains at least two vertices in each copy of H_5. On the other hand, $\{v_3, v_4\}$ is an irreversible 2-conversion set of H_5. It follows that by attaching a copy of H_5 to a vertex of degree 1, the minimum size of an irreversible 2-conversion set grows by 1. □

If G_2 is 3-regular, we may directly apply the result of Ueno, Kajitani and Gotoh [25]. Now we take care of the case when G_2 has exactly one or two vertices of degree 2. First we consider the special case when the two vertices of degree 2 are connected by an edge. We subdivide this edge by a new vertex x and add one more vertex y joined to x, forming a graph G_1. See Figure 7.

Lemma 12. *Suppose that u and v are the only vertices of degree 2 in G_2, and that uv is an edge of G_2. Let G_1 be the graph $(V(G_2) \cup \{x, y\}, (E(G_2) \setminus \{uv\}) \cup \{ux, vx, xy\})$. Then $C_2(G_1) = C_2(G_2) + 1$.*

Proof. If S is an irreversible 2-conversion set in G_2, then S contains at least one of the vertices u, v. We claim that $S \cup \{y\}$ is an irreversible 2-conversion set in G_1. This is clear if both u and v are in S. If exactly one of the vertices u, v is in S, say, $u \in S$ and $v \notin S$, then x turns black in the first step. Therefore, it is sufficient to show that $S \cup \{x, y\}$ is an irreversible 2-conversion set in G_1. But this follows since in this case, the irreversible 2-conversion process on the subset $V(G_2)$ is identical to the process on G_2 starting with S black.

Conversely, let S' be an irreversible 2-conversion set in G_1. Then necessarily $y \in S'$. We may assume that $x \notin S'$, otherwise we may replace x by u or v, or remove x from S' if both u and v are in S'. We claim that $S' \setminus \{y\}$ is an irreversible 2-conversion set in G_2. Clearly, at least one of the vertices u, v, say, u, is in S'. If also $v \in S'$, the claim follows immediately. If $v \notin S'$, then during the irreversible 2-conversion process on G_1, the vertex v turns black only after its neighbor in G_2 other than u turns black. Therefore, the irreversible 2-conversion process on G_2 starting with $S' \setminus \{y\}$ black will be induced by the process on G_1 starting with S' black. □

Modifying G_1 like in Lemma 11, that is, by attaching a copy of H_5 to y, we obtain a graph G_2' with two nonadjacent vertices of degree 2 and all other vertices of degree 3, satisfying $C_2(G_2') = C_2(G_1) + 1 = C_2(G_2) + 2$.

Now we consider the case of two nonadjacent vertices of degree 2.

Lemma 13. *Suppose that u and v are the only vertices of degree 2 in G_2 and that u and v are not adjacent. Then the graph G_3 obtained from G_2 by adding the edge uv satisfies $C_2(G_3) = C_2(G_2)$. In particular, every irreversible 2-conversion set in G_3 is also an irreversible 2-conversion set in G_2.*

Proof. The inequality $C_2(G_2) \geq C_2(G_3)$ is trivial. For the other inequality, suppose that S is an irreversible 2-conversion set in G_3. We show that then S is also an irreversible 2-conversion set in G_2. Every component of $G_3 - S$ is a tree. In the beginning, the vertices of S are black and the other vertices are white. In each step, the irreversible 2-threshold process converts all isolated vertices and all leaves of the white subgraph of G_3 to black vertices. If w is an isolated vertex in $G_3 - S$, w has still at least two black neighbors in G_2, so it is converted to a black vertex in the first step.

Let T be a tree component T of $G_3 - S$ with at least two vertices. If uv is an edge of T, the two components of $T - uv$ will still be converted to black vertices in G_2, with u and v being the last vertices to be converted. If $u \in T$ and $v \notin T$, then all vertices of T will still be converted to black vertices, with u being the last vertex to be converted. □

We note that we could use Lemma 13 also in the case when uv is an edge, if we allowed multigraphs. However, we have decided not to use multigraphs in this paper.

The case of exactly one vertex of degree 2 can be easily solved using Lemma 13 by taking two disjoint copies of G_2.

Corollary 14. *Suppose that v is the only vertex of degree 2 in G_2. Then the graph G_3 obtained from G_2 by adding a disjoint graph G_2' isomorphic to G_2 and joining the vertex v' of degree 2 in G_2' with v by an edge is 3-regular and satisfies $C_2(G_3) = 2C_2(G_2)$.*

We are left with the case when G_2 has at least $k \geq 3$ vertices of degree 2. In this case, we construct a 3-regular graph G_3 by attaching a caterpillar T with k leaves and $k-2$ vertices of degree 3 forming the spine; see Figure 8. Every leaf of T is identified with one vertex of degree 2 in G_2. Let V_2 be the vertex set of G_2 and let V_3 be the vertex set of G_3. The graph $G_3 - V_2$ is a path induced by the $k-2$ branching nodes of T.

Let $\mu(G)$ be the *cyclomatic number* of G. That is, $\mu(G) = e(G) - v(G) + \kappa(G)$, where $e(G)$, $v(G)$ and $\kappa(G)$ are the numbers of edges, vertices and components of G, respectively.

Define a function $f : 2^{V_3} \to \mathbb{Z}$ from the set of subsets of vertices of G_3 as $f(X) := \mu(G_3) - \mu(G_3 - X)$. Roughly speaking, f measures the number of cycles broken by X in G_3. Let $f_2 : 2^{V_2} \to \mathbb{Z}$ be the restriction of f to subsets of V_2. Ueno, Kajitani and Gotoh [25] proved that (V_3, f) is a linear 2-polymatroid, using a linear representation of the dual matroid of the graphic matroid of G_3. More precisely, each vertex v of G_3 can be represented as a 2-dimensional subspace $h(v)$ of a certain vector space (over any field, and of dimension not exceeding the number of vertices of G_3) so that $f(X)$ is equal to the dimension of the span

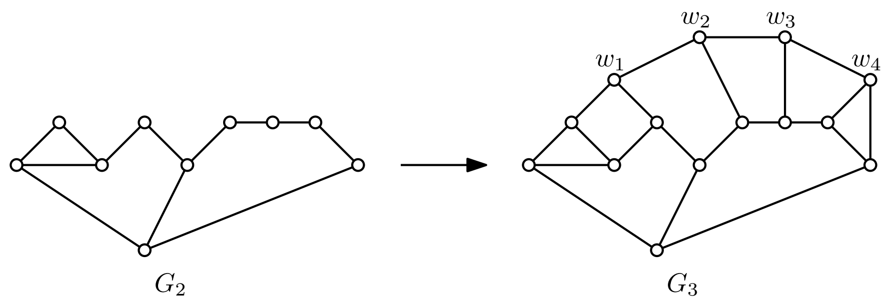


Figure 8: Extending G_2 to a 3-regular graph G_3 by attaching a caterpillar.

of $\bigcup\{h(v); v \in X\}$. The function f is called the *rank function* of the 2-polymatroid. Since f_2 is a restriction of f, it follows that (V_2, f_2) is also a linear 2-polymatroid.

A set M is a *matching* in a 2-polymatroid with rank function f if $f(M) = 2|M|$. A set S is *spanning* in a 2-polymatroid (V, f) if $f(S) = f(V)$. Let $\nu(V, f)$ be the maximum size of a matching in (V, f) and let $\rho(V, f)$ be the minimum size of a spanning set of (V, f). Lovász [18, 19] proved the following generalization of Gallai's identity.

Lemma 15 ([18],[19, Lemma 11.1.1.]). *For every 2-polymatroid (V, f), we have $\nu(V, f) + \rho(V, f) = f(V)$.*

Lovász [18] proved that a maximum matching in a linear 2-polymatroid can be found in polynomial time. It follows that also $\rho(V, f)$ can be computed in polynomial time, for any linear 2-polymatroid (V, f).

The theorem now follows from the following fact, generalizing [25, Theorem 3].

Lemma 16. *A set $S \subseteq V_2$ is a spanning set in (V_2, f_2) if and only if it is an irreversible 2-conversion set in G_2.*

To prove the lemma, we use the following simple observation.

Observation 17. *Let S be an irreversible 2-conversion set in a graph G. Let v be a vertex of G of degree 2 such that $v \notin S$. Then S is an irreversible 2-conversion set in $G - v$.* □

Proof of Lemma 16. Let $S \subseteq V_2$ be an irreversible 2-conversion set in G_2. We claim that S is also an irreversible 2-conversion set in G_3. If not, then $G_3 - S$ contains a cycle C of white vertices that will not be converted to black vertices during the irreversible 2-threshold process starting with S black. Since $G_3 - V_2$ is a tree, C contains a vertex of V_2; a contradiction. Therefore, S is a feedback vertex set in G_3, equivalently, $G_3 - S$ is acyclic, and this is equivalent to the fact that $f(S) = f(V_3)$. Since $G_3 - V_2$ is acyclic, we have $f_2(S) = f(S) = f(V_3) = f(V_2) = f_2(V_2)$, and so S is spanning in (V_2, f_2).

Now let $S \subseteq V_2$ be a spanning set in (V_2, f_2). By the previous arguments, this is equivalent to the fact that S is an irreversible 2-conversion set in G_3. Let $w_1 w_2 \dots w_{k-2}$ be the path $G_3 - V_2$. Let e be an edge joining w_{k-2} with a vertex of V_2. By Lemma 13, S is an irreversible 2-conversion set in $G_3 - e$. By Observation 17, S is an irreversible 2-conversion set in $(G_3 - e) - w_{k-2} = G_3 - w_{k-2}$. Similarly, by a repeated application of Observation 17, S is an irreversible 2-conversion set in $G_3 - w_{k-2} - w_{k-3} - \cdots - w_1 = G_2$. □

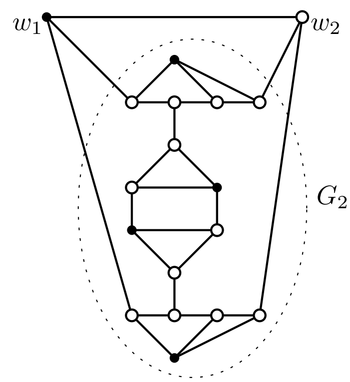


Figure 9: A 3-regular graph G_3 with an irreversible 2-conversion set of size 5. Every such set has to contain at least one of the vertices w_1, w_2; therefore, the connected subgraph $G_2 = G_3 - w_1 - w_2$ has no irreversible 2-conversion set of size 5.

3.1 Running time of the algorithm

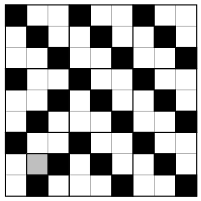
Lovász and Plummer [19] estimated the running time of Lovász's algorithm [18] to be $O(n^{17})$. Since every 2-dimensional subspace can be represented by a pair of linearly independent vectors, the matching problem for 2-polymatroids is equivalent to the *linear matroid parity problem*, whose input is a linear matroid with a partition of its edges into pairs, and the goal is to find an independent set with maximum number of pairs. For matroids with n elements and rank r, Gabow and Stallmann [15, 16] gave an algorithm for the linear matroid parity problem running in time $O(nr^{\omega})$, where $O(n^{\omega})$ is the complexity of multiplication of two square $n \times n$ matrices. Moreover, for graphic matroids, Gabow and Stallmann [15, 16] gave a very fast algorithm running in time $O(nr \log^6 r)$. They noted that the same algorithm can be used to solve the linear matroid parity problem for cographic matroids, that is, the duals of graphic matroids. This follows from the simple fact that a maximum matching M and a basis B containing M determine a unique maximum matching in the dual matroid in the complement of B. Takaoka, Tayu and Ueno [24] used Gabow's and Stallmann's algorithm to show that the vertex feedback set problem for graphs of maximum degree 3 can be solved in time $O(n^2 \log^6 n)$. However, we are not able to make a similar conclusion for the I2CS problem, and can guarantee only $O(n^{1+\omega})$ running time using the more general algorithm by Gabow and Stallmann.

Although the matroid (V_3, f) constructed by Ueno, Kajitani and Gotoh [25] is cographic, our submatroid (V_2, f_2) is not cographic in general. Moreover, $\rho(V_3, f)$ can be smaller than $\rho(V_2, f_2)$; see Figure 9. Therefore, we cannot directly use the value $\nu(V_3, f)$ computed by the faster algorithm by Gabow and Stallmann.

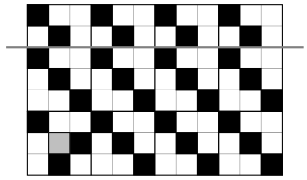
Problem 1. *Can the I2CS problem on graphs of maximum degree* 3 *be efficiently reduced to the cographic matroid parity problem?*

4 Irreversible 3-conversion set in toroidal grids

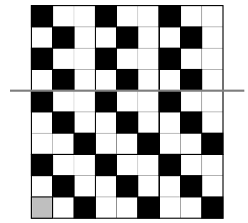
In this section we show a construction of an irreversible 3-conversion set S that proves Theorem 3. We denote the toroidal grid of size $n \times m$ by $T(n, m)$. When the dimensions of the



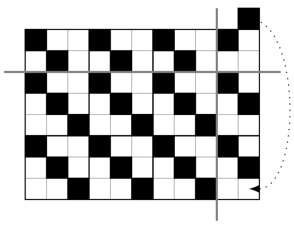
(A) $3k, 3l$



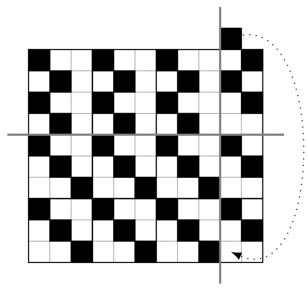
(B) $3k, 3l+2$



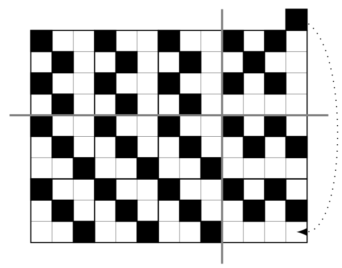
(C) $3k, 3l+4$



(D) $3k+2, 3l+2$



(E) $3k+2, 3l+4$



(F) $3k+4, 3l+4$

Figure 10: The cases for $T(m, n)$.

grid are clear from the context or not important, we simply write T instead of $T(m, n)$. We assume that the entries of the grid are *squares* and two of them are neighboring if they share an edge. First we discuss the general case where $m \neq 4$ and $n \neq 4$.

We define a coordinate system on T such that the left bottom corner is $[0, 0]$. A *pattern* is a small and usually rectangular piece of a grid where squares are black and white. *Placing a pattern P at position* $[i, j]$ in T means that the left bottom square of P is at $[i, j]$ in T. If a vertex of T has color defined by several patterns then it is white only if it is white in all the patterns. We describe a rectangle of a grid by the coordinates of the bottom left corners of its bottom left and top right squares. *Tiling* a rectangle R by a pattern P means placing several non-overlapping copies of P to R so that every square of R is covered.

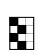
Let $m = 3k+a$ and $n = 3l+b$, where $a, b \in \{0, 2, 4\}$. By g we denote the greatest common divisor of k and l.

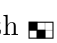
For $0 \leq i \leq g-2$ we place a pattern  at $[0, 3i]$. Next we tile the rest of the rectangle $[0, 0][3k-1, 3l-1]$ by a pattern . The remaining part of the grid can be decomposed into three rectangles of dimensions $3k \times b$, $a \times 3l$ and $a \times b$ (some of them may be empty).

We distinguish several cases depending on a and b. They are depicted in Figure 10 and their description follows.

(A) $\boldsymbol{a = 0, b = 0}$ We do not add anything now.

(B) $\boldsymbol{a = 0, b = 2}$ We tile the rectangle $3k \times 2$ with .

(C) $\boldsymbol{a = 0, b = 4}$ We tile the rectangle $3k \times 4$ with .

(D) $\boldsymbol{a = 2, b = 2}$ We tile the rectangle $3k \times 2$ with , the rectangle $2 \times 3l$ with  and place  at $[3k, 3l]$.

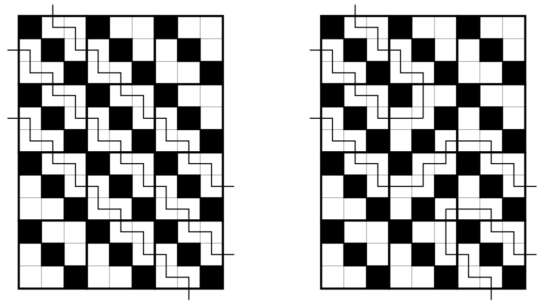
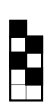
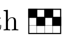
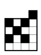


Figure 11: Merging three white cycles into one.

(E) $a = 2, b = 4$ We tile the rectangle $3k \times 4$ with , the rectangle $2 \times 3l$ with  and place  at $[3k, 3l]$.

(F) $a = 4, b = 4$ We tile the rectangle $3k \times 4$ with , the rectangle $4 \times 3l$ with  and place  at $[3k, 3l]$.

The construction is finished for cases (D), (E) and (F). Cases (A), (B) and (C) require an extra black square. We place it at [0,0] or [1,1]. It is colored grey in Figure 10.

Let S be the set of black squares in our construction. In the cases (A), (B) and (C) the size of S is $\frac{mn}{3} + 1 = \frac{mn+3}{3}$. In the cases (D) and (F) the size of S is $\frac{mn+2}{3}$ and in the last case (E) the size of S is $\frac{mn+4}{3}$.

Now we check the correctness of the construction. We start with the case (A) where $m = 3k$ and $n = 3l$.

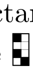
By a *white cycle* we denote a connected set of white squares $W \subseteq T$ where every square in W has at least two neighbors in W. Note that T cannot contain any white cycle if the squares of an irreversible 3-conversion set are black.

Observation 18. *Let $T(3k, 3l)$ be filled with* *. Then it contains g disjoint white cycles.*

Let the whole grid be filled by . By Observation 18, there are g white cycles after the filling. The idea of our construction is to merge the cycles into one long cycle by changing  to  in the first column and the first $g - 1$ rows; see Figure 11. Finally, we add one more black vertex to break the resulting cycle.

Observe that the small patterns used in (B) – (F) just extend the size of the toroidal grid but do not change the structure of white cycles from the $3k \times 3l$ rectangle. Thus the argument for the case (A) can be easily extended to all the other cases.

This finishes the construction for the general case.

Now we assume without loss of generality that $n = 4$. Let $m = 2k + a$, where $a \in \{1, 2\}$. We tile the rectangle $[0, 0][2k - 1, 3]$ by a pattern . If $a = 1$ we place  at $[2k - 1, 0]$ and if $a = 2$ we place  at $[2k, 0]$. The resulting grids are depicted in Figure 12.

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$2k, 4$ $\quad$ $2k+1, 4$

Figure 12: The cases for $T(m, 4)$.

to the papers containing results on minimum irreversible k-conversion sets in high dimensional grids.

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