# Sidorenko property and forcing in regular tournaments*

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**Abstract**

We give a complete characterization of tournaments $H$ that have the Sidorenko property with respect to nearly regular tournaments, i.e., the homomorphism density of $H$ among all nearly regular tournaments is minimized by a random tournament. Corollaries of our result are a positive answer to the question of Noel, Ranganathan and Simbaqueba whether there exist infinitely many non-transitive tournaments that are quasirandom forcing for nearly regular tournaments, and a negative answer to their question whether almost every tournament is quasirandom forcing for nearly regular tournaments.

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# 1 Introduction

The work presented in this paper is motivated by problems concerning quasirandomness of tournaments (orientations of complete graphs). Informally speaking, a combinatorial structure is said to be *quasirandom* if it has properties that a random structure would have asymptotically almost surely. The study of *quasirandom graphs* can be traced back to the nowadays classical works of Rödl [42], Thomason [49, 50] and Chung, Graham and Wilson [13] from 1980s. The notion of quasirandom graphs is particularly robust as there are seemingly different characterizations of quasirandom graphs, such as through homomorphism counts, the distribution of edges, the cut sizes, algebraic properties, etc., and so it has found applications in many different settings. There is a long series of results concerning quasirandomness of other kinds of combinatorial structures, for example groups [27], hypergraphs [9, 10, 25, 26, 32, 33, 39, 43], permutations [4, 19, 34, 35], tournaments [3, 11, 21, 29, 31, 40], subsets of integers [12], etc.

We are interested in *quasirandom forcing* substructures. We illustrate this property on quasirandom graphs, likely the most studied notion of quasirandom structures. A graph $H$ is quasirandom forcing if the following holds for every sequence $(G_n)_{n\in\mathbb{N}}$ of graphs: the sequence $(G_n)_{n\in\mathbb{N}}$ is quasirandom if and only if the limit of the homomorphism density of $H$ in $(G_n)_{n\in\mathbb{N}}$ is equal to the expected homomorphism density of $H$ in a random graph. In other words, a graph is quasirandom if and only if the homomorphism density of $H$ is close to its expected density, i.e., any non-randomness necessarily results in the deviation from the expected homomorphism density of $H$. Examples of quasirandom forcing graphs include even cycles and complete bipartite graphs with each part of size at least two.

The concept of quasirandom forcing is intimately related to the notion of Sidorenko graphs. A graph $H$ has the *Sidorenko property* if the homomorphism density of $H$ is asymptotically minimized by a random graph. One of the most intriguing questions in extremal combinatorics is a conjecture of Sidorenko [44] and of Erdős and Simonovits [22] that asserts that every bipartite graph has the Sidorenko property; we refer particularly to [2, 14, 16–18, 45, 46] for classes of bipartite graphs proven to have the Sidorenko property. It is easy to show that every quasirandom forcing graph must have the Sidorenko property, and the Forcing Conjecture of Conlon, Fox and Sudakov [14], a well-known generalization of the above mentioned conjecture based on a question of Skokan and Thoma [47], is equivalent to the statement that a graph $H$ is quasirandom forcing if and only if $H$ is bipartite and has at least one cycle.

Our work is motivated by quasirandom forcing in the setting of tournaments. It is well-known that transitive tournaments have the Sidorenko property, i.e., their homomorphism density is asymptotically minimized by a random tournament, and every transitive tournament with with at least four vertices is quasirandom forcing, see [21] and [38, Exercise 10.44]. Coregliano, Parente and Sato [20]

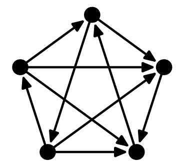

Figure 1: The unique quasirandom forcing tournament that is not transitive.

identified another quasirandom forcing tournament on five vertices, which is depicted in Figure 1. Surprisingly, this 5-vertex tournament has the anti-Sidorenko property, i.e., its homomorphism density is asymptotically maximized by a random tournament. Bucić, Long, Shapira and Sudakov [3] observed that there are no additional quasirandom forcing tournaments with seven or more vertices, and the remaining tournaments on at most six vertices were analyzed by Hancock et al. [31]. Hence, the transitive tournaments with at least four vertices and the 5-vertex tournament depicted in Figure 1 are the only quasirandom forcing tournaments, which is in contrast with the setting of graphs where very rich families of quasirandom forcing graphs are known [14, 16–18].

Noel, Ranganathan and Simbaqueba [40] considered quasirandom forcing in the setting of nearly regular tournaments. A sequence $(T_n)_{n\in\mathbb{N}}$ of tournaments is *nearly regular* if for every $\varepsilon > 0$, the proportion of vertices in $T_n$ whose outdegree differ from $|V(T_n)|/2$ by more than $\varepsilon|V(T_n)|$ tends to zero. Clearly, if a tournament $H$ is quasirandom forcing for all sequences of tournaments, then $H$ is quasirandom forcing for nearly regular sequences. In the other direction, if $H$ is quasirandom forcing for nearly regular sequences of tournaments, then $\{H, H'\}$ is a quasirandom forcing family for all sequences where $H'$ is any tournament forcing near regularity, for example $H'$ can be chosen to be the cyclically oriented triangle or the 3-vertex transitive tournament.

Noel, Ranganathan and Simbaqueba [40] characterized all tournaments with at most five vertices that are quasirandom forcing for nearly regular sequences of tournaments. In particular, they identified three such additional 4-vertex tournaments and five 5-vertex tournaments. Furthermore, they posed the following three open problems, out of which we answer two and we resolve the remaining one up to finitely many cases.

**Problem 1** (Noel, Ranganathan and Simbaqueba [40, Problem 6.1])**.** *Characterize tournaments that are quasirandom forcing for nearly regular sequences of tournaments.*

**Problem 2** (Noel, Ranganathan and Simbaqueba [40, Question 6.2])**.** *Are there infinitely many non-transitive tournaments that are quasirandom forcing for nearly regular sequences of tournaments?*

**Problem 3** (Noel, Ranganathan and Simbaqueba [40, Question 6.3])**.** *Is almost every tournament quasirandom forcing for nearly regular sequences of tournaments?*

Our main result, which we state as Corollary 10, gives a full list of tournaments $H$ that have the Sidorenko property for nearly regular tournaments. There are two types of such tournaments: *transitive tournaments* and *blow-ups of the cyclically oriented triangle such that each part of the blow-up induces a transitive tournament* (such blow-ups are depicted in Figure 4). In addition, we obtain a characterization of tournaments $H$ (Corollary 11) that are quasirandom forcing in addition to having the Sidorenko property in this setting. In particular, we identify infinitely many such non-transitive tournaments and so we answer the question given in Problem 2 in the affirmative. Note that every quasirandom forcing tournament must have either the Sidorenko property or the anti-Sidorenko property. As we observe that no tournament with at least ten vertices has the anti-Sidorenko property (Proposition 12 in Section 5), any additional tournament that is quasirandom forcing for nearly regular sequences of tournaments must have at most nine vertices. Hence, we obtain a negative answer to Problem 3 and solve Problem 1 up to finitely many cases.

## 1.1 Sketch of the main argument

Before proceeding with presenting our arguments, we would like to briefly highlight the main steps and ideas. We treat the problem in the language of combinatorial limits, which we introduce in Section 2; we refer the reader to this section for notions used in this paragraph that has not yet been defined. In Section 3, we show that if $H$ is neither a transitive tournament nor a blow-up of the cyclically oriented triangle as in the characterization, then there are regular tournaments that are $H$-free. The core arguments are presented in Section 4. An easy argument (formally given in Lemma 8) yields that it is enough to show that the blow-ups of the cyclically oriented triangle with independent parts, which are denoted by $C[a,b,c]$ (see Figure 3), are quasirandom forcing for nearly regular tournaments.

Let us now sketch the proof for the case $a = b$, which we believe to transparently capture the main idea. A key observation is that $t(C[1,1,c],W) = t(B[c],W)/2$ for any regular tournamenton $W$ where $B[c]$ is the digraph obtained from $C[1,1,c]$ by removing the edge between the parts of size one (see Figure 5). We next define three auxiliary functions (the formal definitions are given before Lemma 6): $N^+_{W,c} : [0,1]^c \to [0,1]$ that measures the size of the "common out-neighborhood" of a $c$-tuple of points, $N^-_{W,c} : [0,1]^c \to [0,1]$ that measures the size of the "common in-neighborhood", and $D_{W,c} : [0,1]^c \to [0,1]$ that measures the density of edges from the "common out-neighborhood" to the "common in-neighborhood". We immediately obtain that

$$t(B[c],W) = \int N^-_{W,c}(x) N^+_{W,c}(x)\,\mathrm{d}x_{[c]} \qquad \text{and}$$
$$t(C[1,1,c],W) = \int N^-_{W,c}(x) D_{W,c}(x) N^+_{W,c}(x)\,\mathrm{d}x_{[c]}.$$

On the other hand, the Sidorenko property of complete bipartite graphs oriented from one part to another yields that

$$t(C[a,a,c],W) \geq \int N^-_{W,c}(x)^a D_{W,c}(x)^{a^2} N^+_{W,c}(x)^a\,\mathrm{d}x_{[c]}. \tag{1}$$

Jensen's Inequality and Hölder Inequality now imply that

$$t(C[1,1,c],W) \leq t(B[c],W)^{\frac{a-1}{a}} t(C[a,a,c],W)^{\frac{1}{a^2}}.$$

In general, it is hard to bound $t(B[c],W)$ in estimates as the one above (an analogous issue has prevented us from extending the entropy proof presented in Section 5 to all values of $a$, $b$ and $c$), however, as we pointed out, it holds that $t(C[1,1,c],W) = t(B[c],W)/2$ for regular tournamentons $W$. Hence, we obtain that

$$2^{-a^2}\, t(B[c],W)^a \leq t(C[a,a,c],W);$$

this estimate yields the result as $B[c]$ can be shown to have the Sidorenko property and to be quasirandom forcing by standard arguments. The actual proof of Theorem 7 consists of a generalization of the Sidorenko type inequality (1) and careful applications of Jensen's Inequality and Hölder's Inequality.

# 2 Preliminaries

In this section, we overview the notation used throughout the paper. We first start with an overview of the general notation that we use and is less standard. We write $[k]$ for the set $\{1,\ldots,k\}$. We also use $\mathbb{Z}_k$ for the set $[k]$ when the additional algebraic structure given by the addition modulo $k$ is of importance. In general, all integrals are over the space $[0,1]^k$ with Lebesgue measure unless specified otherwise. Finally, we write $x_A$ for $x \in \mathbb{R}^A$, i.e., a vector whose coordinates are indexed by the elements of $A$. Using the just introduced notation, $x_{[k]}$ is a vector $x \in \mathbb{R}^k$ and we will write $x_{[k]}$ instead of simply writing $x$ when we wish to emphasize the dimension of the vector $x$. For example,

$$\int x_1 x_2\,\mathrm{d}x_{[2]} = \frac{1}{4}.$$

We next introduce the notation related to digraphs and tournaments. All digraphs considered in this paper are simple, i.e., without loops and parallel

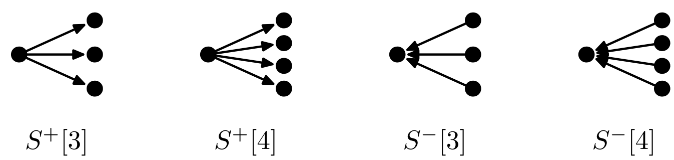


Figure 2: The digraphs $S^+[3]$, $S^+[4]$, $S^-[3]$ and $S^-[4]$.

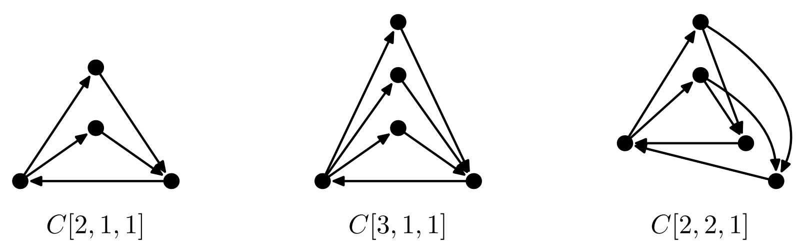


Figure 3: The digraphs $C[2,1,1]$, $C[3,1,1]$ and $C[2,2,1]$, which are blow-ups of the cyclically oriented triangle with parts of the sizes given by the parameters.

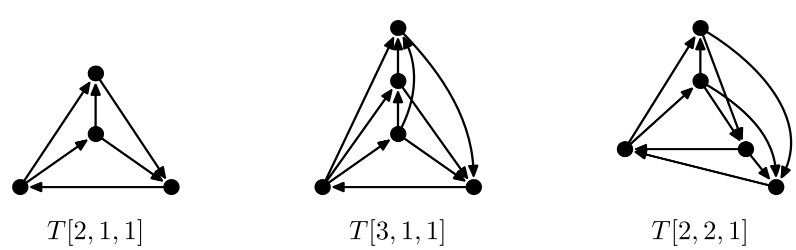


Figure 4: The tournaments $T[2,1,1]$, $T[3,1,1]$ and $T[2,2,1]$.

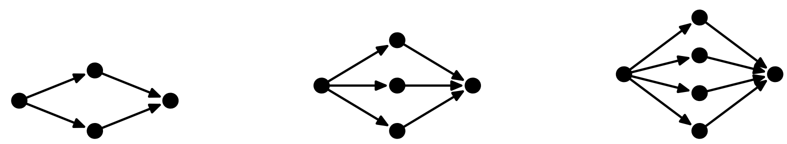


Figure 5: The digraphs $B[2]$, $B[3]$ and $B[4]$.

edges. The vertices $u$ and $v$ of a digraph are *twins* if the out-neighbors of $u$ are exactly the out-neighbors of $v$, the in-neighbors of $u$ are exactly the in-neighbors of $v$, and there is no edge between $u$ and $v$. We write $S^+[k]$ and $S^-[k]$, where $k \in \mathbb{N}$, for the orientation of the $k$-leaf star such that the central vertex is the source and the sink, respectively; see Figure 2 for illustrations. Fix $a, b, c \in \mathbb{N}$. The digraph $C[a, b, c]$ is the blow-up of the cyclically oriented triangle with the parts of sizes $a$, $b$ and $c$ respectively, i.e., $C[a, b, c]$ has three parts, one with $a$ vertices, one with $b$ vertices and one with $c$ vertices, and it contains all the edges between the $a$-vertex and the $b$-vertex parts directed to the $b$-vertex part, all the edges between the $b$-vertex and the $c$-vertex parts directed to the $c$-vertex part, and all the edges between the $a$-vertex and the $c$-vertex parts directed to the $a$-vertex part; see Figure 3 for examples. Note that all pairs of vertices contained in the same of the three parts of $C[a, b, c]$ are twins. The tournament $T[a, b, c]$ is obtained from the digraph $C[a, b, c]$ by adding a transitive tournament on the vertices of each of the three parts of the digraph $C[a, b, c]$; see Figure 4 for examples. Finally, recall the digraph $B[c]$ is obtained from $C[1, 1, c]$ by removing the edge between the two vertices contained in the parts of size one; see Figure 5. Note that $B[c]$ can be viewed as obtained from $c$ directed paths of length two by identifying their first vertices to a single (source) vertex and their last vertices to a single (sink) vertex.

We next introduce the necessary concepts from the theory of combinatorial limits and refer for a more thorough introduction to [31,40], where these tools were used in the context of quasirandom forcing tournaments. A tournamenton is an analytic object that represents a convergent sequence of tournaments; informally speaking, tournamentons can be thought of as adjacency matrices of large tournaments. Formally, a *tournamenton* is a measurable function $W : [0, 1]^2 \to [0, 1]$ such that $W(x, y) + W(y, x) = 1$ for all $(x, y) \in [0, 1]^2$. Quasirandom sequences of tournaments are represented by the constant tournamenton, i.e., the tournament $W$ such that $W(x, y) = 1/2$ for all $(x, y) \in [0, 1]^2$. Throughout the paper, we will use a shorthand notation $W \equiv 1/2$ to represent that a tournamenton $W$ is equal to $1/2$ almost everywhere. A tournamenton $W$ is *regular* if

$$\int W(x, y)\,\mathrm{d}y = \frac{1}{2}$$

for almost every $x \in [0,1]$; loosely speaking, regular tournamentons represent large tournaments where every vertex has asymptotically the same in-degree and out-degree. Formally, regular tournamentons are limits of nearly regular convergent sequences of tournaments.

Let $H$ be a digraph. The *homomorphism density* of $H$ in a tournamenton $W$, denoted by $t(H,W)$ is defined as follows:

$$t(H,W) = \int \prod_{vw \in E(H)} W(x_v, x_w)\,\mathrm{d}x_{V(H)}. \tag{2}$$

We write $t(H,1/2)$ for the homomorphism density of $H$ in the constant tournamenton.

A digraph $H$ has the *Sidorenko property* if $t(H,W) \geq 2^{-|E(H)|}$ for every tournamenton $W$ and $H$ has the *anti-Sidorenko property* if $t(H,W) \leq 2^{-|E(H)|}$ for every tournamenton $W$. A digraph $H$ is *quasirandom forcing* if a tournamenton $W$ satisfies that $t(H,W) = 2^{-|E(H)|}$ if and only if $W \equiv 1/2$. Similarly, $H$ is *quasirandom forcing in regular tournamentons* if a regular tournamenton $W$ satisfies that $t(H,W) = 2^{-|E(H)|}$ if and only if $W \equiv 1/2$.

We next cast several classical results concerning tournaments in the language of combinatorial limits. The first concerns the homomorphism density of transitive tournaments.

**Proposition 1.** *Let $n \in \mathbb{N}$ and let $T$ be the $n$-vertex transitive tournament. For every tournamenton $W$, it holds that*

$$t(T,W) \geq 2^{-\binom{n}{2}}.$$

*Moreover, the equality holds if and only if*

- $n \in \{1,2\}$ *and $W$ is arbitrary,*
- $n = 3$ *and $W$ is regular, or*
- $n \geq 4$ *and $W \equiv 1/2$.*

Since there are only two 3-vertex tournaments, which are the transitive 3-vertex tournament and the cyclically oriented triangle, we derive from Proposition 1 the following.

**Proposition 2.** *Every tournamenton $W$ satisfies that $t(C[1,1,1],W) \leq 1/8$ and the equality holds if and only if $W$ is regular.*

We finish this section with a generalization of Hölder's Inequality, which will be used later. Let $\Omega$ be a probability space with the probability measure $\mu$ and let

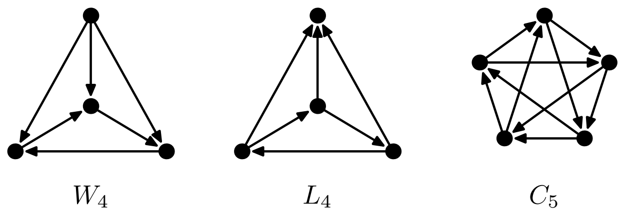


Figure 6: The tournaments $W_4$, $L_4$ and $C_5$.

$k$ be a positive integer. For any collection of measurable functions $F_i : \Omega \to [0, 1]$, $i \in [k]$, it holds that

$$\int_\Omega \prod_{i\in[k]} F_i(x)\,\mathrm{d}\mu(x) \le \prod_{i\in[k]} \left(\int_\Omega F_i(x)^{p_i}\,\mathrm{d}\mu(x)\right)^{1/p_i}$$

whenever $p_1, \ldots, p_k$ are non-negative reals such that $1/p_1 + \cdots + 1/p_k \le 1$.

# 3 Constructions

In this section, we present regular tournamentons witnessing that a tournament $T$ does not have the Sidorenko property for nearly regular tournaments unless $T$ is a transitive tournament or a tournament $T[a, b, c]$ for some $a, b, c \in \mathbb{N}$. A key step in this argument is the following lemma giving a structural characterization of tournaments of the latter kind. To state the lemma, we introduce the following notation (see Figure 6 for illustration): $W_4$ is the 4-vertex tournament with one vertex being a source and the remaining vertices forming a cyclically oriented triangle, $L_4$ is the 4-vertex tournament with one vertex being a sink and the remaining vertices forming a cyclically oriented triangle, and $C_5$ is the 5-vertex carousel tournament, i.e., the vertex of $C_5$ can be viewed as $\mathbb{Z}_5$ and $uv$ is an edge if and only if $v - u \equiv 1 \mod 5$ or $v - u \equiv 2 \mod 5$.

**Lemma 3.** *Every non-transitive tournament that contains neither of $W_4$, $L_4$ and $C_5$ is isomorphic to $T[a, b, c]$ for some $a, b, c \in \mathbb{N}$.*

*Proof.* Fix $T$ a non-transitive tournament that contains neither of the tournaments $L_4$, $W_4$ and $C_5$. Since $T$ is not transitive, it contains a cyclically oriented triangle; let $u_1u_2u_3$ be any cyclically oriented triangle of $T$.

We now partition the vertices $V$ into three sets $V_1$, $V_2$ and $V_3$ as follows. First, for every $i \in [3]$, the vertex $u_i$ is included to the set $V_i$. Let $v$ be a vertex of $V$ different from $u_1$, $u_2$ and $u_3$. Since the tournament $T$ contains neither $W_4$ nor $L_4$, the vertex $v$ has both an in-neighbor and an out-neighbor among the vertices $u_1$, $u_2$ and $u_3$. If the vertex $v$ has exactly one in-neighbor among these three

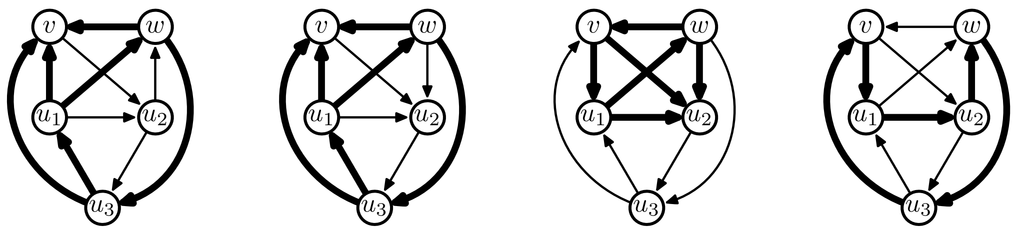


Figure 7: The 5-vertex tournament induced by vertices $u_i$, $u_{i+1}$, $u_{i+2}$, $v$ and $v'$ that depends on the direction of the edge between the vertices $u_i$ and $v$ and the edge between $u_{i+1}$ and $v'$. The figure displays the case $i = 1$ (other cases are symmetric). The subgraphs isomorphic to $W_4$ or $L_4$ are depicted by bold edges in the first three cases and the edges of one of the cycles of length five in $C_5$ are depicted by bold edges in the fourth case.

vertices, say $u_i$, we add $v$ to the set $V_{i+1}$ (the index is taken modulo 3). If the vertex $v$ has exactly two in-neighbors among the vertices $u_1$, $u_2$ and $u_3$, then $v$ has exactly one out-neighbor among them, say $u_j$, we add $v$ to the set $V_{j-1}$ (again, the index is taken modulo 3). We conclude that the set $V_i$ is non-empty for every $i \in [3]$ ($V_i$ contains the vertex $u_i$), and each vertex of the set $V_i$ has $u_{i-1}$ among its in-neighbors and $u_{i+1}$ among its out-neighbors.

Fix $i \in [3]$. Consider any vertex $v$ contained in $V_i$ different from $u_i$ and any vertex $w$ contained in $V_{i+1}$ different from $u_{i+1}$ (indices modulo 3). In case the edge between $v$ and $w$ was directed from $w$ to $v$, the tournament $T$ would contain one of the tournaments $L_4$, $W_4$ and $C_5$ (the four cases that depend on the direction of the edges $u_iv$ and $u_{i+1}w$ are drawn in Figure 7). Hence, the edge between $v$ and $w$ is directed from $v$ to $w$. Since the choice of $v$ and $w$ was arbitrary, we conclude that all the edges of $T$ between $V_i$ and $V_{i+1}$ are directed from $V_i$ to $V_{i+1}$ (indices modulo 3).

Finally, note that for every $i \in V_i$, every triple of vertices of $V_i$ induces a transitive tournament (otherwise, the triple and the vertex $u_{i+1}$ would form the tournament $W_4$). Hence, the tournaments induced by each of the sets $V_1$, $V_2$ and $V_3$ are transitive, and $T$ is isomorphic to the tournament $T[|V_1|, |V_2|, |V_3|]$. □

We are now ready to prove the main theorem of this section.

**Theorem 4.** *Let $T$ be a tournament that is neither a transitive tournament nor a tournament $T[a, b, c]$ for some $a, b, c \in \mathbb{N}$. Then, there exists a regular tournamenton $W$ such that $t(T, W) = 0$.*

*Proof.* By Lemma 3,it is enough to establish the theorem for $T$ being $W_4$, $L_4$ and $C_5$. Rather than exhibiting a specific regular tournamenton, we will construct a sequence $(T_n)_{n\in\mathbb{N}}$ of regular tournaments such that $t(T, T_n) = 0$ and the number of vertices of $T_n$ tends to infinity. The sought tournamenton $W$ will be a limit tournamenton of a convergent subsequence of $(T_n)_{n\in\mathbb{N}}$; we remark that the sequences that we construct are actually convergent, however, we do not need this stronger claim.

Let $T_n$ be the $(2n+1)$-vertex carousel tournament, i.e., the vertices of $T_n$ are $\mathbb{Z}_{2n+1}$ and there is an edge directed from $x$ to $y$ if and only if $y - x \in \{1, \dots, n\}$ (modulo $2n+1$). Note that the out-neighbors of any vertex of $T_n$ induce a transitive tournament and likewise the in-neighbors of any vertex induce a transitive tournament. It follows that $t(W_4, T_n) = 0$ and $t(L_4, T_n) = 0$, which establishes the statement when $T$ is $W_4$ or $L_4$.

It remains to consider the case $T = C_5$. Let $T_n$ be the $n$-th iterated blow-up of the cyclically oriented triangle, i.e., the vertices of $T_n$ are $\mathbb{Z}_3^n$ and there is an edge directed from $x$ to $y$ if and only if $y_i - x_i = 1$ (modulo 3) for the smallest index $i$ such that $x_i \neq y_i$. We claim that $t(C_5, T_n) = 0$. Suppose that there are five vertices $v^1, \dots, v^5$ of $T_n$ inducing $C_5$ and let $i$ be the smallest index such that two of the vertices differ in the $i$-th coordinate. By symmetry, we may assume $v_i^1 = 1$ and $v_i^2 = 2$. One of the vertices must have the $i$-th coordinate equal to 3 (otherwise, the induced tournament is not strongly connected); hence, we can also assume that $v_i^3 = 3$. By symmetry, we may assume that $v_i^4 = 1$ and $v_i^5 \in \{1, 2\}$. If $v_i^5 = 1$, then the in-degree of $v_2$ is three, which is impossible. If $v_i^5 = 2$, then both $v^1$ and $v^4$ are in-neighbors of each of the vertices $v^2$ and $v^5$, and since there is an edge $v^2v^5$, the in-degree of either $v^2$ or $v^5$ is three, which is not possible. We conclude that no five vertices of $T_n$ induce $C_5$ and so $t(C_5, T_n) = 0$. $\square$

# 4 Main result

In this section, we complement the results presented in Section 3 by showing that every tournament $T[a, b, c]$ with $a + b + c \geq 4$ is quasirandom forcing in regular tournamentons. We start with the following auxiliary lemma on digraphs $B[k]$; recall that $B[k]$ is the digraph obtained from $C[1, 1, k]$ by removing the edge between the two vertices contained in the parts of size one.

**Lemma 5.** *Let $k \geq 2$. Every regular tournamenton $W$ satisfies that $t(B[k], W) \geq 2^{-2k}$ and the equality holds if and only if $W \equiv 1/2$.*

*Proof.* Fix $k \geq 2$ and a regular tournamenton $W$. We define two auxiliary functions $F : [0,1]^2 \to [0,1]$ and $G : [0,1]^2 \to [0,1]$ as follows:

$$F(x, y) = \int W(x, z)W(y, z)\,\mathrm{d}z \text{ and}$$
$$G(x, y) = \int W(x, z)W(z, y)\,\mathrm{d}z.$$

Informally speaking, $F(x, y)$ measures the number of common out-neighbors of $x$ and $y$ and $G(x, y)$ measures the number of paths of length two from $x$ to $y$.

Observe that it holds for every $x, y \in [0,1]$ that

$$F(x,y)+G(x,y) = \int W(x,z)W(y,z)+W(x,z)W(z,y)\,\mathrm{d}z = \int W(x,z)\,\mathrm{d}z = \frac{1}{2}. \tag{3}$$

Since the tournamenton $W$ is regular, it holds that

$$\int F(x,y)\,\mathrm{d}x\,\mathrm{d}y = t(S^-[2],W) = \frac{1}{4},$$

which implies using (3) that

$$\int G(x,y)\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{4}.$$

Jensen's Inequality now yields that

$$t(B[k],W) = \int G(x,y)^k\,\mathrm{d}x\,\mathrm{d}y \geq \left(\int G(x,y)\,\mathrm{d}x\,\mathrm{d}y\right)^k = 2^{-2k},$$

and the equality holds if and only if $G(x,y) = 1/4$ for almost all $(x,y) \in [0,1]^2$. By (3), it holds that $t(B[k],W) = 2^{-2k}$ if and only if $F(x,y) = 1/4$ for almost all $(x,y) \in [0,1]^2$. However, this is exactly the limit formulation of the property $P_4$ of quasirandom sequences of tournaments from [11]. □

To state the next lemma, we need to define three auxiliary functions, which will be parameterized by $k \in \mathbb{N}$. The functions $N^+_{W,k} : [0,1]^k \to [0,1]$ and $N^-_{W,k} : [0,1]^k \to [0,1]$ measure the "size" of the common out-neighborhood and the common in-neighborhood of a $k$-tuple points in a tournamenton $W$:

$$N^+_{W,k}(x_1,\ldots,x_k) = \int \prod_{i\in[k]} W(x_i,z)\,\mathrm{d}z \text{ and}$$

$$N^-_{W,k}(x_1,\ldots,x_k) = \int \prod_{i\in[k]} W(z,x_i)\,\mathrm{d}z.$$

Finally, the function $D_{W,k} : [0,1]^k \to [0,1]$ measures the density of edges directed from the common out-neighborhood to the common in-neighborhood:

$$D_{W,k}(x_1,\ldots,x_k) = \frac{\int W(y,z) \prod\limits_{i\in[k]} W(x_i,y)W(z,x_i)\,\mathrm{d}y\,\mathrm{d}z}{N^+_{W,k}(x_1,\ldots,x_k)N^-_{W,k}(x_1,\ldots,x_k)}.$$

When the tournamenton $W$ is clear from context, we drop it from the subscript and simply write $N^+_k$, $N^-_k$ and $D_k$. We will also write $x_{[k]}$ instead of $x_1,\ldots,x_k$.

Observe that the following holds for every $k \in \mathbb{N}$ and every tournamenton $W$:

$$t\left(C[1,1,k],W)\right) = \int N^+_{W,k}(x_{[k]})D_{W,k}(x_{[k]})N^-_{W,k}(x_{[k]})\,\mathrm{d}x_{[k]}. \tag{4}$$

The next lemma says that the digraphs $C[a,b,k]$ have a Sidorenko type property:

**Lemma 6.** *The following holds for every tournamenton $W$ and all $a,b,k \in \mathbb{N}$:*

$$t\left(C[a,b,k],W\right) \geq \int N^+_{W,k}(x_{[k]})^a D_{W,k}(x_{[k]})^{ab} N^-_{W,k}(x_{[k]})^b\,\mathrm{d}x_{[k]} \tag{5}$$

*Proof.* The proof of the lemma follows the standard argument for the Sidorenko property of bipartite graphs. Throughout the proof, we use $W(x_{[k]},y_{[\ell]})$ as a shorthand notation for the double product $\prod_{i\in[k]}\prod_{j\in[\ell]} W(x_i,y_j)$ where $x \in [0,1]^k$ and $y \in [0,1]^\ell$. We extend the notation to the case when $k=1$ or $\ell=1$, e.g., $W(x_{[k]},y)$ stands for the product $\prod_{i\in[k]} W(x_i,y)$.

Fix $a,b,k \in \mathbb{N}$ and a tournamenton $W$ for the proof. Consider $x \in [0,1]^k$ such that $N^+_k(x_{[k]}) > 0$ and $N^-_k(x_{[k]}) > 0$. Observe that it holds that

$$\int \frac{W(x_{[k]},y_{[a]})}{N^+_k(x_{[k]})^a}\,\mathrm{d}y_{[a]} = \prod_{j\in[a]} \int \frac{W(x_{[k]},y_j)}{N^+_k(x_{[k]})}\,\mathrm{d}y_j = 1.$$

In particular, we can interpret the integrand in the left integral as the density of a probability measure on $[0,1]^a$. We now apply Jensen's Inequality as follows:

$$\begin{aligned}
&\int \frac{W(x_{[k]},y_{[a]})W(y_{[a]},z_{[b]})W(z_{[b]},x_{[k]})}{N^+_k(x_{[k]})^a N^-_k(x_{[k]})^b}\,\mathrm{d}y_{[a]}\,\mathrm{d}z_{[b]} \\
= &\int \frac{W(x_{[k]},y_{[a]})}{N^+_k(x_{[k]})^a} \int \frac{W(y_{[a]},z_{[b]})W(z_{[b]},x_{[k]})}{N^-_k(x_{[k]})^b}\,\mathrm{d}z_{[b]}\,\mathrm{d}y_{[a]} \\
= &\int \frac{W(x_{[k]},y_{[a]})}{N^+_k(x_{[k]})^a} \left(\int \frac{W(y_{[a]},z)W(z,x_{[k]})}{N^-_k(x_{[k]})}\,\mathrm{d}z\right)^b \mathrm{d}y_{[a]} \\
\geq &\left(\int \frac{W(x_{[k]},y_{[a]})}{N^+_k(x_{[k]})^a} \int \frac{W(y_{[a]},z)W(z,x_{[k]})}{N^-_k(x_{[k]})}\,\mathrm{d}z\,\mathrm{d}y_{[a]}\right)^b \\
= &\left(\int \frac{W(x_{[k]},y_{[a]})W(y_{[a]},z)W(z,x_{[k]})}{N^+_k(x_{[k]})^a N^-_k(x_{[k]})}\,\mathrm{d}y_{[a]}\,\mathrm{d}z\right)^b.
\end{aligned} \tag{6}$$

Next observe that it holds that

$$\int \frac{W(z,x_{[k]})}{N^-_k(x_{[k]})}\,\mathrm{d}z = 1,$$

which means that we can interpret the integrand as the density of a probability measure on $[0,1]$. So, we get by another application of Jensen's Inequality the following:

$$\begin{aligned}
&\int \frac{W(x_{[k]},y_{[a]})W(y_{[a]},z)W(z,x_{[k]})}{N_k^+(x_{[k]})^a N_k^-(x_{[k]})}\,\mathrm{d}y_{[a]}\,\mathrm{d}z \\
= &\int \frac{W(z,x_{[k]})}{N_k^-(x_{[k]})}\left(\int \frac{W(x_{[k]},y)W(y,z)}{N_k^+(x_{[k]})}\,\mathrm{d}y\right)^a \mathrm{d}z \\
\geq &\left(\int \frac{W(z,x_{[k]})}{N_k^-(x_{[k]})}\int \frac{W(x_{[k]},y)W(y,z)}{N_k^+(x_{[k]})}\,\mathrm{d}y\,\mathrm{d}z\right)^a \\
= &\left(\int \frac{W(x_{[k]},y)W(y,z)W(z,x_{[k]})}{N_k^+(x_{[k]})N_k^-(x_{[k]})}\,\mathrm{d}y\,\mathrm{d}z\right)^a = D_k(x_{[k]})^a. \qquad (7)
\end{aligned}$$

Let $\Omega \subseteq [0,1]^k$ be the set of those $x \in [0,1]^k$ such that $N_k^+(x_{[k]}) > 0$ and $N_k^-(x_{[k]}) > 0$. Using (6) and (7) we conclude that for every $x \in \Omega$ it holds that

$$\begin{aligned}
&\int W(x_{[k]},y_{[a]})W(y_{[a]},z_{[b]})W(z_{[b]},x_{[k]})\,\mathrm{d}y_{[a]}\,\mathrm{d}z_{[b]} \\
= \;& N_k^+(x_{[k]})^a N_k^-(x_{[k]})^b \int \frac{W(x_{[k]},y_{[a]})W(y_{[a]},z_{[b]})W(z_{[b]},x_{[k]})}{N_k^+(x_{[k]})^a N_k^-(x_{[k]})^b}\,\mathrm{d}y_{[a]}\,\mathrm{d}z_{[b]} \\
\geq \;& N_k^+(x_{[k]})^a N_k^-(x_{[k]})^b \left(\int \frac{W(x_{[k]},y_{[a]})W(y_{[a]},z)W(z,x_{[k]})}{N_k^+(x_{[k]})^a N_k^-(x_{[k]})}\,\mathrm{d}y_{[a]}\,\mathrm{d}z\right)^b \\
\geq \;& N_k^+(x_{[k]})^a N_k^-(x_{[k]})^b D_k(x_{[k]})^{ab}. \qquad (8)
\end{aligned}$$

Using that inequality (8) holds for every $x \in \Omega$, we estimate the homomorphism density of $C[a,b,k]$ as follows:

$$\begin{aligned}
t\left(C[a,b,k],W\right) &= \int W(x_{[k]},y_{[a]})W(y_{[a]},z_{[b]})W(z_{[b]},x_{[k]})\,\mathrm{d}x_{[k]}\,\mathrm{d}y_{[a]}\,\mathrm{d}z_{[b]} \\
&\geq \int_\Omega W(x_{[k]},y_{[a]})W(y_{[a]},z_{[b]})W(z_{[b]},x_{[k]})\,\mathrm{d}x_{[k]}\,\mathrm{d}y_{[a]}\,\mathrm{d}z_{[b]} \\
&\geq \int_\Omega N_k^+(x_{[k]})^a D_k(x_{[k]})^{ab} N_k^-(x_{[k]})^b\,\mathrm{d}x_{[k]}.
\end{aligned}$$

Since it holds that $N_k^+(x_{[k]})^a D_k(x_{[k]})^{ab} N_k^-(x_{[k]})^b = 0$ for every $x \in [0,1]^k \setminus \Omega$, we obtain that

$$t\left(C[a,b,k],W\right) \geq \int N_k^+(x_{[k]})^a D_k(x_{[k]})^{ab} N_k^-(x_{[k]})^b\,\mathrm{d}x_{[k]}.$$

Since the choice of $a,b,k \in \mathbb{N}$ and a tournamenton $W$ was arbitrary, the proof of the lemma is completed. □

We are now ready to prove the key theorem of this section, which we then use to prove Theorem 9.

**Theorem 7.** *Let $a, b, c \in \mathbb{N}$ such that $a+b+c \geq 4$. Every regular tournamenton $W$ satisfies that*

$$t\left(C[a,b,c], W\right) \geq 2^{-ab-ac-bc}. \tag{9}$$

*Moreover, the equality in* (9) *holds if and only if $W \equiv 1/2$.*

*Proof.* Fix $a, b, c \in \mathbb{N}$ such that $a+b+c \geq 4$ and a regular tournamenton $W$. Note that

$$t\left(C[a,b,c], W\right) = t\left(C[b,c,a], W\right) = t\left(C[c,a,b], W\right);$$

so we can assume by this rotational symmetry that $a$ is the smallest among $a$, $b$ and $c$, and if the smallest value is not unique among $a$, $b$ and $c$, then it additionally holds that $a = b$. Observe that $c \geq 2$ (if $c = 1$, then $a = b = 1$, which is impossible as $a+b+c \geq 4$).

As in Lemma 5, let $G : [0,1]^2 \to [0,1]$ be defined as

$$G(x,y) = \int W(x,z)W(z,y)\,\mathrm{d}z;$$

informally speaking, $G(x,y)$ is the density of directed paths from $x$ to $y$ of length two. Also note that $G$ is the square of $W$ in the operator sense. We observe that the regularity of $W$ implies that the function $G$ is symmetric, i.e., it holds that

$$\begin{aligned}
G(y,x) &= \int W(y,z)W(z,x)\,\mathrm{d}z \\
&= \int (1-W(z,y))(1-W(x,z))\,\mathrm{d}z \\
&= 1 - \int W(x,z)\,\mathrm{d}z - \int W(z,y)\,\mathrm{d}z + \int W(x,z)W(z,y)\,\mathrm{d}z \\
&= 1 - \frac{1}{2} - \frac{1}{2} + \int W(x,z)W(z,y)\,\mathrm{d}z = G(x,y)
\end{aligned}$$

Recall that $B[c]$ is the digraph obtained from $C[1,1,c]$ by removing the edge between the two vertices contained in the parts of size one. The definition of $G$ yields that

$$\begin{aligned}
t\left(B[c], W\right) &= \int G(x,y)^c\,\mathrm{d}x\,\mathrm{d}y \text{ and} \\
t\left(C[1,1,c], W\right) &= \int G(x,y)^c W(y,x)\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Since the function $G(x,y)$ is symmetric and $W(x,y)+W(y,x)=1$ for all $(x,y)\in[0,1]^2$, we obtain that

$$\begin{aligned}
t\left(C[1,1,c],W\right) &= \frac{1}{2}t\left(C[1,1,c],W\right)+\frac{1}{2}t\left(C[1,1,c],W\right)\\
&= \frac{1}{2}\int G(x,y)^cW(y,x)\,\mathrm{d}x\,\mathrm{d}y+\frac{1}{2}\int G(y,x)^cW(x,y)\,\mathrm{d}x\,\mathrm{d}y\\
&= \frac{1}{2}\int G(x,y)^cW(y,x)+G(y,x)^cW(x,y)\,\mathrm{d}x\,\mathrm{d}y\\
&= \frac{1}{2}\int G(x,y)^cW(y,x)+G(x,y)^cW(x,y)\,\mathrm{d}x\,\mathrm{d}y\\
&= \frac{1}{2}\int G(x,y)^c\,\mathrm{d}x\,\mathrm{d}y=\frac{t\left(B[c],W\right)}{2}. \qquad (10)
\end{aligned}$$

Recall the definitions of the functions $N_c^+:[0,1]^c\to[0,1]$, $N_c^-:[0,1]^c\to[0,1]$ and $D_c:[0,1]^c\to[0,1]$ given before the statement of Lemma 6. Also recall that $S^+[c]$ is the directed $c$-leaf star with the center vertex being the source and $S^-[c]$ is the directed $c$-leaf star with the center vertex being the sink. Observe that the following four identities hold (we use that $W$ is regular in the last two):

$$\begin{aligned}
t\left(B[c],W\right) &= \int N_c^+(x_{[c]})N_c^-(x_{[c]})\,\mathrm{d}x_{[c]}, && (11)\\
t\left(C[1,1,c],W\right) &= \int N_c^+(x_{[c]})D_c(x_{[c]})N_c^-(x_{[c]})\,\mathrm{d}x_{[c]}, && (12)\\
t\left(S^+[c],W\right) &= 2^{-c}=\int N_c^-(x_{[c]})\,\mathrm{d}x_{[c]}\text{ and} && (13)\\
t\left(S^-[c],W\right) &= 2^{-c}=\int N_c^+(x_{[c]})\,\mathrm{d}x_{[c]}. && (14)
\end{aligned}$$

We now apply generalized Hölder's Inequality with $p_1=ab$, $p_2=\frac{ab}{ab-b}$ and $p_3=\frac{ab}{b-a}$ (recall that $a\le b$) to obtain the following inequality:

$$\begin{aligned}
\int N_c^+(x_{[c]})D_c(x_{[c]})N_c^-(x_{[c]})\,\mathrm{d}x_{[c]} \le{}& \left(\int N_c^+(x_{[c]})^aD_c(x_{[c]})^{ab}N_c^-(x_{[c]})^b\,\mathrm{d}x_{[c]}\right)^{\frac{1}{ab}}\times\\
&\left(\int N_c^+(x_{[c]})N_c^-(x_{[c]})\,\mathrm{d}x\right)^{\frac{ab-b}{ab}}\times\\
&\left(\int N_c^+(x_{[c]})\,\mathrm{d}x\right)^{\frac{b-a}{ab}}. \qquad (15)
\end{aligned}$$

We plug (11), (12) and (14) to (15) and get that

$$\begin{aligned}
t\left(C[1,1,c],W\right) \le{}& \left(\int N_c^+(x_{[c]})^aD_c(x_{[c]})^{ab}N_c^-(x_{[c]})^b\,\mathrm{d}x_{[c]}\right)^{\frac{1}{ab}}\times\\
&t\left(B[c],W\right)^{\frac{ab-b}{ab}}\times 2^{\frac{-c(b-a)}{ab}},
\end{aligned}$$

which yields by Lemma 6 that

$$t\left(C[1,1,c],W\right) \le t\left(C[a,b,c],W\right)^{\frac{1}{ab}} \times t\left(B[c],W\right)^{\frac{ab-b}{ab}} \times 2^{\frac{-c(b-a)}{ab}}. \tag{16}$$

The inequality (16) is equivalent to

$$t\left(C[a,b,c],W\right) \ge \frac{t\left(C[1,1,c],W\right)^{ab}}{2^{-c(b-a)} \times t\left(B[c],W\right)^{ab-b}}. \tag{17}$$

Using (10) and Lemma 5, we manipulate the right side of (17) as follows:

$$\frac{t\left(C[1,1,c],W\right)^{ab}}{2^{-c(b-a)} \times t\left(B[c],W\right)^{ab-b}} = \frac{2^{-ab} \times t\left(B[c],W\right)^{b}}{2^{ac-bc}} \ge \frac{2^{-ab-2bc}}{2^{ac-bc}} = 2^{-ac-ab-bc};$$

note that Lemma 5 yields that the equality above holds if and only if $W \equiv 1/2$. We conclude that

$$2^{-ac-ab-bc} \le t\left(C[a,b,k],W\right)$$

and the equality holds if and only if $W \equiv 1/2$. □

Before proving our main theorem, we need an auxiliary lemma. Informally speaking, we use that transitive tournaments have the Sidorenko property to show that adding a transitive tournament on a vertex set formed by twins in a digraph $H$ drops density of $H$ by at most the expected homomorphism density of the added transitive tournament in a random tournament.

**Lemma 8.** *Let $H$ be a digraph that contains an independent set $A$ such that all vertices in $A$ are twins, and let $H'$ be the digraph obtained from $H$ by adding the $|A|$-vertex transitive tournament on $A$. It holds that*

$$t(H',W) \ge 2^{-\binom{|A|}{2}}\, t(H,W).$$

*for every tournamenton $W$.*

*Proof.* The proof will proceed by induction on the size of the set $A$. Before presenting the proof, we introduce some notation. Consider a digraph $H$, an independent set $A$ as in the statement of the lemma and let $a \in A$ be a vertex of $A$; as the vertices of $A$ are twins, what follows is independent of this choice. Let $B$ be the set of the remaining vertices of $H$. For a tournamenton $W$, we define two functions $F_W : [0,1]^B \times [0,1] \to [0,1]$ and $G_W : [0,1]^B \to [0,1]$ as follows:

$$F_W(x_B,z) = \prod_{va \in E(H)\cap(B\times\{a\})} W(x_v,z) \prod_{av \in E(H)\cap(\{a\}\times B)} W(z,x_v) \text{ and}$$
$$G_W(x_B) = \prod_{uv \in E(H)\cap(B\times B)} W(x_u,x_v).$$

Observe that

$$t(H,W) = \int G_W(x_B) \prod_{a\in A} F_W(x_B, z_a) \, \mathrm{d}x_B \, \mathrm{d}z_A$$

for any tournamenton $W$.

We are now ready to present the proof of the lemma, in which we use the above introduced notation.

If $|A| = 1$, then the statement holds trivially as $H = H'$. We next analyze the case $|A| = 2$. The definition of $H'$ implies that

$$t(H',W) = \int G_W(x_B)F_W(x_B, z_1)F_W(x_B, z_2)W(z_1, z_2) \, \mathrm{d}x_B \, \mathrm{d}z_1 \, \mathrm{d}z_2.$$

Since the role of $z_1$ and $z_2$ in the above expression is symmetric we also have that

$$t(H',W) = \int G_W(x_B)F_W(x_B, z_1)F_W(x_B, z_2)W(z_2, z_1) \, \mathrm{d}x_B \, \mathrm{d}z_1 \, \mathrm{d}z_2,$$

which yields using the identity $W(z_1, z_2) + W(z_2, z_1) = 1$ for all $(z_1, z_2) \in [0,1]^2$ that

$$2t(H',W) = \int G_W(x_B)F_W(x_B, z_1)F_W(x_B, z_2) \, \mathrm{d}x_B \, \mathrm{d}z_1 \, \mathrm{d}z_2 = t(H,W).$$

This concludes the proof of the case $|A| = 2$. Note that we have proven that the inequality always holds with equality when $|A| = 2$.

We now establish the induction step in the case $|A| \geq 3$. Let $a_1, \ldots, a_{|A|}$ be the vertices of $A$ listed in the order that is consistent with the transitive tournament on $A$ in $H'$, and let $H''$ be the digraph obtained from $H$ by adding an edge directed from $a_1$ to each of the vertices $a_2, \ldots, a_{|A|}$. Note that the vertices of $A \setminus \{a_1\}$ are twins in $H''$. By the induction hypothesis applied to the digraph $H''$ with the digraph $H'$ and the independent set $A \setminus \{a_1\}$, we obtain that

$$t(H',W) \geq 2^{-\binom{|A|-1}{2}} \, t(H'',W). \tag{18}$$

We next apply the induction to the digraph $H$ and the digraph obtained from $H$ by adding the edge directed from $a_1$ to $a_2$, i.e., we invoke the case when the size of the independent set is two and the inequality holds with equality to obtain that

$$\int G_W(x_B) \left( \prod_{a\in A} F_W(x_B, z_a) \right) W(z_{a_1}, z_{a_2}) \, \mathrm{d}x_B \, \mathrm{d}z_A = \frac{t(H,W)}{2}.$$

We now apply Hölder's Inequality to derive that

$$\frac{t(H,W)}{2} = \int G_W(x_B) \left( \prod_{a \in A} F_W(x_B, z_a) \right) W(z_{a_1}, z_{a_2}) \, \mathrm{d}x_B \, \mathrm{d}z_A$$

$$= \int G_W(x_B) F_W(x_B, z_1) \left( \int F_W(x_B, z) W(z_1, z) \, \mathrm{d}z \right) \left( \int F_W(x_B, z) \, \mathrm{d}z \right)^{|A|-2} \mathrm{d}x_B \, \mathrm{d}z_1$$

$$\leq \left( \int G_W(x_B) F_W(x_B, z_1) \left( \int F_W(x_B, z) W(z_1, z) \, \mathrm{d}z \right)^{|A|-1} \mathrm{d}x_B \, \mathrm{d}z_1 \right)^{\frac{1}{|A|-1}} \times$$

$$\left( \int G_W(x_B) F_W(x_B, z_1) \left( \int F_W(x_B, z) \, \mathrm{d}z \right)^{|A|-1} \mathrm{d}x_B \, \mathrm{d}z_1 \right)^{\frac{|A|-2}{|A|-1}}$$

$$= t(H'', W)^{\frac{1}{|A|-1}} \times t(H, W)^{\frac{|A|-2}{|A|-1}},$$

which yields that

$$2^{-|A|+1} \, t(H,W) \leq t(H'', W). \tag{19}$$

The inequalities (18) and (19) combine to

$$t(H', W) \geq 2^{-\binom{|A|-1}{2} - |A| + 1} \, t(H, W) = 2^{-\binom{|A|}{2}} \, t(H, W).$$

This completes the induction step. □

We are now ready to prove the main theorem of the paper.

**Theorem 9.** *Let $a, b, c \in \mathbb{N}$ such that $a + b + c \geq 4$. Every regular tournamenton $W$ satisfies that*

$$t(T[a,b,c], W) \geq 2^{-\binom{a+b+c}{2}},$$

*and the equality holds if and only if $W \equiv 1/2$.*

*Proof.* Fix $a, b, c \in \mathbb{N}$ such that $a + b + c \geq 4$ and a regular tournamenton $W$. If $W \equiv 1/2$, then $t(T[a,b,c], W) = 2^{-\binom{a+b+c}{2}}$. We assume that $W \not\equiv 1/2$ for the rest of the proof and show that $t(T[a,b,c], W) > 2^{-\binom{a+b+c}{2}}$. Theorem 7 implies that

$$t(C[a,b,c], W) > 2^{-ab-ac-bc}. \tag{20}$$

Let $T_1$ be the digraph obtained from $C[a,b,c]$ by adding the $a$-vertex transitive tournament on the part of size $a$ and let $T_2$ be the digraph obtained from $T_1$ by adding the $b$-vertex transitive tournament on the part of size $b$; observe that $T[a,b,c]$ is the digraph obtained from $T_2$ by adding the $c$-vertex transitive tournament on the part of size $c$. Lemma 8 yields the following inequalities:

$$t(T_1, W) \geq 2^{-\binom{a}{2}} \, t(C[a,b,c], W),$$

$$t(T_2, W) \geq 2^{-\binom{b}{2}} \, t(T_1, W) \text{ and}$$

$$t(T[a,b,c], W) \geq 2^{-\binom{c}{2}} \, t(T_2, W).$$

We now combine these three inequalities with (20) to obtain that

$$t(T[a,b,c],W) \geq 2^{-\binom{a}{2}-\binom{b}{2}-\binom{c}{2}}\, t(C[a,b,c],W)$$
$$> 2^{-\binom{a}{2}-\binom{b}{2}-\binom{c}{2}-ab-ac-bc} = 2^{-\binom{a+b+c}{2}},$$

which completes the proof of the theorem. □

Theorem 9 yields the following two corollaries. The first gives a classification of tournaments with the Sidorenko property for nearly regular tournaments and the second gives a classification of such tournaments that are quasirandom forcing.

**Corollary 10.** *Let $T$ be a tournament. It holds that $t(T,W) \geq t(T,1/2) = 2^{-\binom{|V(T)|}{2}}$ for every regular tournamenton $W$ if and only if $T$ is a transitive tournament or a tournament $T[a,b,c]$ for some $a,b,c \in \mathbb{N}$.*

*Proof.* If $T$ is neither a transitive tournament nor a tournament $T[a,b,c]$ for some $a,b,c \in \mathbb{N}$, then there exists a regular tournamenton $W$ such that $t(T,W) = 0$ by Theorem 4. If $T$ is a transitive tournament, then the inequality follows from Proposition 1. Hence, we can assume that $T$ is the graph $T[a,b,c]$ for some $a,b,c \in \mathbb{N}$. If $a = b = c = 1$, i.e., $T$ is a cyclically oriented triangle, then the inequality hold with equality by Proposition 2, and if $a+b+c \geq 4$, then the inequality follows from Theorem 9. □

**Corollary 11.** *Let $T$ be a tournament. The unique minimizer of $t(T,W)$ among all regular tournamentons $W$ is the constant tournamenton if and only if the tournament $T$ is a transitive tournament with at least four vertices or a tournament $T[a,b,c]$ for some $a,b,c \in \mathbb{N}$ such that $a+b+c \geq 4$.*

*Proof.* Consider a tournament $T$. Corollary 10 yields that the constant tournamenton is a minimizer of $t(T,W)$ among all regular tournamentons $W$ if and only if $T$ is a transitive tournament or a tournament $T[a,b,c]$ for some $a,b,c \in \mathbb{N}$. If $T$ is a transitive tournament, then Proposition 1 yields that the constant tournamenton is the unique minimizer of $t(T,W)$ if and only if $T$ has at least four vertices. Hence, we can assume that $T$ is the graph $T[a,b,c]$ for some $a,b,c \in \mathbb{N}$. If $a = b = c = 1$, then all regular tournamentons $W$ satisfy that $t(T,W) = 1/8$ and so the constant tournamenton is not the unique minimizer of $t(T,W)$, and if $a+b+c \geq 4$, then the constant tournamenton is the unique minimizer by Theorem 9. □

# 5 Conclusion

The entropy method has recently gained a prominent role in making progress on various problems in extremal combinatorics, particularly Sidorenko type problems, see e.g. [1, 5–8, 15–17, 23, 30, 36, 37, 41, 48]. We found a proof of Theorem 7 using the entropy method when $c = 1$ and $a$ and $b$ are arbitrary but we were unable to extend it to full generality. Still, we want to sketch the argument. We will assume that the reader is familiar with basic concepts concerning the use of the entropy method and we refer to e.g. [24, 28] for the exposition in the setting of combinatorics.

Fix $a, b \in \mathbb{N}$ and let $G$ be an $n$-vertex regular tournament. It is well-known that $G$ has $\frac{n^3-n}{24} = \frac{1}{4}\binom{n}{3} + O(n^2)$ cyclically oriented triangles and every vertex of $G$ is in exactly $\frac{n^2-1}{8}$ of such triangles, i.e., every vertex is in the same number of cyclically oriented triangles. We will estimate the number of homomorphisms from $T[a, b, 1]$ to $G$. Let $(x, y, z)$ be a cyclically oriented triangle of $G$ chosen uniformly at random so that $xy$, $yz$ and $zx$ are edges in $G$. Note that we consider $(x, y, z)$ and $(y, z, x)$ to be different as we wish to count homomorphisms. Note that $H(x, y, z) = \log \frac{n^3-n}{8}$ and

$$H(x, y) = H(y, z) = H(z, x) \leq \log \binom{n}{2} \leq \log \frac{n^2}{2};$$

the first inequality in the displayed estimate follows from the fact that entropy is maximized by uniform distribution and as $G$ is a tournament, there are exactly $\binom{n}{2}$ directed edges. We now sample $(y, z)$ according to the marginal distribution coming from $(x, y, z)$ and sample $x_1, \ldots, x_a$ as conditionally independent copies of $x$ given $(y, z)$. In this way, we obtain a distribution on $(a + 2)$-tuples $(x_1, \ldots, x_a, y, z)$ that correspond to homomorphisms from $C[a, 1, 1]$ to $G$. Note that

$$H(x_1, \ldots, x_a, y, z) = aH(x|y, z) + H(y, z). \tag{21}$$

We now sample $(x_1, \ldots, x_a, z)$ according to the marginal distribution coming from the distribution of $(x_1, \ldots, x_a, y, z)$ and sample $y_1, \ldots, y_b$ as conditionally independent copies of $y$ given $(x_1, \ldots, x_a, z)$. In this way, we obtain a distribution on $(a+b+1)$-tuples $(x_1, \ldots, x_a, y_1, \ldots, y_b, z)$ that correspond to homomorphisms from $C[a, b, 1]$ to $G$ and we can compute its entropy as follows:

$$H(x_1, \ldots, x_a, y_1, \ldots, y_b, z) = bH(y|x_1, \ldots, x_a, z) + H(x_1, \ldots, x_a, z). \tag{22}$$

Since the $(a+1)$-tuple $(x_1, \ldots, x_a, z)$ always induces the directed $a$-leaf star with the center vertex being the source and the tournament $G$ is regular, we obtain that

$$H(x_1, \ldots, x_a, z) \leq \log n \left(\frac{n-1}{2}\right)^a \leq \log \frac{n^{a+1}}{2^a}. \tag{23}$$

We now combine (21), (22) and (23) to obtain the following estimate on the entropy of the constructed distribution on $(a+b+1)$-tuples $(x_1, \dots, x_a, y_1, \dots, y_b, z)$:

$$\begin{aligned}
&H(x_1, \dots, x_a, y_1, \dots, y_b, z)\\
&\quad = bH(y|x_1, \dots, x_a, z) + H(x_1, \dots, x_a, z)\\
&\quad = bH(x_1, \dots, x_a, y, z) - (b-1)H(x_1, \dots, x_a, z)\\
&\quad = abH(x|y, z) + bH(y, z) - (b-1)H(x_1, \dots, x_a, z)\\
&\quad = abH(x, y, z) - (a-1)bH(y, z) - (b-1)H(x_1, \dots, x_a, z)\\
&\quad \geq ab\log\frac{n^3-n}{8} - (a-1)b\log\frac{n^2}{2} - (b-1)\log\frac{n^{a+1}}{2^a}\\
&\quad = \log\frac{(n^3-n)^{ab}}{2^{3ab}} - \log\frac{n^{2ab-2b}}{2^{(a-1)b}} - \log\frac{n^{ab-a+b-1}}{2^{a(b-1)}}\\
&\quad = \log\frac{n^{a+b+1} - O(n^{a+b-1})}{2^{ab+a+b}}.
\end{aligned}$$

Comparing the entropy of the constructed distribution with the entropy of the uniform distribution yields that the number homomorphisms from $C[a,b,1]$ to $G$ is at least $\frac{n^{a+b+1}(1+o(1))}{2^{ab+a+b}}$, which implies that $t(C[a,b,1],W) \geq 2^{-ab-a-b}$ for any regular tournamenton $W$; one can also use the entropy method to show that the equality holds if and only if $W \equiv 1/2$.

Recall that [40, Problem 6.1], which was presented as Problem 1 here, asks to characterize tournaments that are quasirandom forcing in regular tournamentons. Note that every quasirandom forcing tournament has either the Sidorenko property or the anti-Sidorenko property. Corollary 11 gives a full characterization of tournaments $T$ such that the constant tournamenton is the unique *minimizer* of $t(T,W)$ among *regular* tournamentons $W$, i.e., all quasirandom forcing tournaments with the Sidorenko property. The 5-vertex tournament depicted in Figure 1 is the only tournament such that the constant tournamenton is the unique *maximizer* of $t(T,W)$ among *all* tournamentons $W$. The next proposition, which uses an argument similar to that from [3], yields that any tournament such that the constant tournamenton is a *maximizer* of $t(T,W)$ among *regular* tournamentons $W$ has at most nine vertices. In particular, Corollary 11 gives a list of all tournaments $T$ that are quasirandom forcing in regular tournamentons except for finitely many cases.

**Proposition 12.** *Let $T$ be an $n$-vertex tournament. If $n \geq 10$, then there exists a regular tournamenton $W$ such that*

$$t(T,W) > 2^{-\binom{n}{2}}.$$

*In particular, there is no tournament $T$ with ten or more vertices such that the constant tournamenton would be a maximizer of the homomorphism density of $T$ among all regular tournamentons.*

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1/2 | 1 | 1 | 1 | 1/2 | 0 | 0 | 0 |
| 0 | 1/2 | 1 | 0 | 1 | 1/2 | 0 | 1 |
| 0 | 0 | 1/2 | 1 | 1 | 1 | 1/2 | 0 |
| 0 | 1 | 0 | 1/2 | 1 | 0 | 1 | 1/2 |
| 1/2 | 0 | 0 | 0 | 1/2 | 1 | 1 | 1 |
| 1 | 1/2 | 0 | 1 | 0 | 1/2 | 1 | 0 |
| 1 | 1 | 1/2 | 0 | 0 | 0 | 1/2 | 1 |
| 1 | 0 | 1 | 1/2 | 0 | 1 | 0 | 1/2 |

Figure 8: The tournamenton $W$ from the proof of Proposition 12 when $T$ is the tournament $W_4$ depicted in Figure 6. The origin of the coordinate system is in the top left corner, the division between the parts as defined in the proof is visualized by dotted lines, and the division between the parts $A_i$'s and $B_i$'s by dashed lines.

*Proof.* Fix a tournament $T$ with $n$ vertices $v_1, \dots, v_n$. We construct a regular tournamenton $W$ that satisfy the inequality given in the proposition. Split the interval $[0,1]$ into $2n$ parts $A_1, \dots, A_n$ and $B_1, \dots, B_n$, each of measure $1/2n$. The tournamenton $W$ is defined as follows:

$$W(x,y) = \begin{cases} 1 & \text{if } x \in A_i,\ y \in A_j \text{ and } v_i v_j \in E(T), \\ 1 & \text{if } x \in B_i,\ y \in B_j \text{ and } v_i v_j \in E(T), \\ 1 & \text{if } x \in A_i,\ y \in B_j \text{ and } v_j v_i \in E(T), \\ 1 & \text{if } x \in B_i,\ y \in A_j \text{ and } v_j v_i \in E(T), \\ 1/2 & \text{if } x \in A_i \cup B_i \text{ and } y \in A_i \cup B_i, \text{ and} \\ 0 & \text{otherwise.} \end{cases}.$$

An example of the tournamenton $W$ when $T$ is the tournament $W_4$ from Figure 6 is given in Figure 8. Observe that if either $x_{v_i} \in A_i$ for all $i \in [n]$ or $x_{v_i} \in B_i$ for all $i \in [n]$, then the integrand in (2) is equal to one. It follows that

$$t(H,W) \geq 2 \cdot (2n)^{-n}.$$

Since $2 \cdot (2n)^{-n} > 2^{-\binom{n}{2}}$ if $n \geq 10$, the statement of the proposition follows. $\square$

We have computationally identified several tournaments $T$ (in addition to those found by Noel, Ranganathan and Simbaqueba [40]) such that the constant

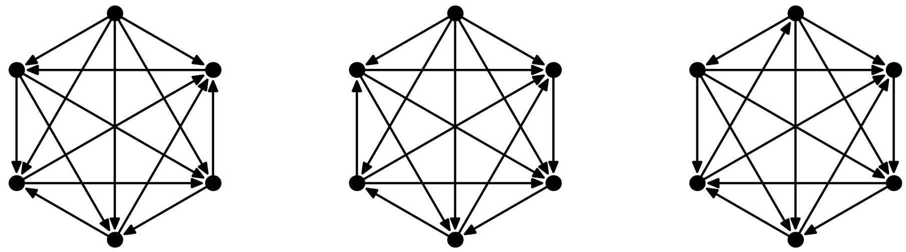

Figure 9: Three 6-vertex tournaments $T$ that we have computationally verified to satisfy that the constant tournamenton is the unique maximizer of $t(T, W)$ among regular tournamentons $W$.

tournamenton is the unique *maximizer* of $t(T, W)$ among *regular* tournamentons $W$; three 6-vertex tournaments $T$ with this property are depicted in Figure 9. We do not have a conjectured list of all such tournaments $T$ and we leave their characterization as an open problem.

**Problem 4.** *Characterize which tournaments $T$ satisfy that the constant tournamenton is the unique maximizer of $t(T, W)$ among regular tournamentons $W$.*

Note that a resolution of Problem 4 together with Corollary 11 would yield a full solution of [40, Problem 6.1] and Proposition 12 asserts that it is enough to investigate tournaments $T$ with at most nine vertices.

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