Density of rainbow triangles and properly colored K_4's

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Abstract

T.-W. Chao and H.-H. H. Yu showed in 2023 that a graph with R red, G green, and B blue edges has at most $\sqrt{2RGB}$ rainbow triangles. They proved this bound using the entropy method. We give a computer-free flag-algebra proof of this bound, and we also convert our proof into a classical counting proof. The ideas in our proof lead to an even shorter entropy proof. We also show uniqueness of the extremal construction.

Additionally, we prove a similar result that gives a sharp upper bound on the number of properly 3-edge-colored K_4's in graphs with R red, G green and B blue edges.

1 Introduction

The following is a classical question in graph theory: Given graphs H_1 and H_2, if G has a fixed number of copies of H_1, what is the maximum number of copies of H_2 that G can have? The earliest instance of this question is Mantel's theorem [15], which states that a triangle-free graph on n vertices contains at most $\frac{1}{4}n^2$ edges. Turán's theorem [18] from 1941 similarly gives an exact answer for the maximum number of edges in a K_r-free graph on n vertices. Zykov's theorem [21] from 1949 generalizes Turán's theorem, showing that Turán's extremal construction also maximizes the number of copies of K_s in a K_r-free graph for each $s < r$.

In 1972, a problem of Erdős and Sós appeared in [9] asking for the maximum number of rainbow triangles in a 3-edge-colored graph on n vertices. They conjectured that the extremal construction is given by the iterated blowup of a properly colored K_4, which contains $(\frac{1}{15} + o(1))n^3$ rainbow triangles (see Figure 1(b)). This conjecture was solved in [2] for n that are powers of 4 or sufficiently large. Their proof used flag algebras and a stability calculation.

Recently, Chao and Hans Yu [5, 6] used the method of entropy to establish another sharp upper bound on the number rainbow triangles. Unlike the result above, which is expressed in terms of the number of vertices, their upper bound is a function of the number of red, green and blue edges.

Theorem 1.1 (Chao and Hans Yu [5])**.** *Let $\Gamma = (V, E)$ be a simple graph, and let the edges of Γ be colored with red, green, and blue. Let R, G, B denote the number of red, green, blue edges, respectively, and T the number of rainbow triangles in Γ. Then, $T^2 \leq 2RGB$.*

While working on [19], Zhao asked the first and the last author if the multiplicative constant 2 in Theorem 1.1 is best possible.[1] Improving on the initial numeric evidence from flag algebras, in this paper

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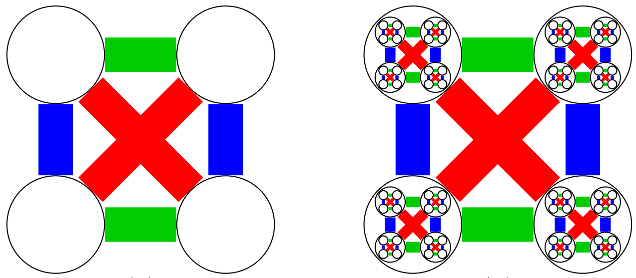
[1] He asked before it became a theorem, i.e., before [5].



(a) (b)

Figure 1: (a) A blowup of a properly 3-edge-colored K_4. (b) An iterated blowup of a properly 3-edge-colored K_4.

we present a proof of Theorem 1.1 using flag algebras, along with an elementary counting proof and a new entropy proof. The latter two proofs are obtained by using parts of our flag algebra argument. We also prove uniqueness of the extremal construction.

Theorem 1.2. *Let Γ be a graph with R red edges, G green edges, and B blue edges. If Γ has $\sqrt{2RGB} > 0$ rainbow triangles, then Γ is obtained from a blowup of a properly 3-edge-colored K_4 by possibly adding a set of isolated vertices.*

For a blowup of a properly 3-edge-colored K_4, see Figure 1(a). We also investigate an analogous problem, where instead of the number of triangles, we are interested in the number of properly 3-edge-colored K_4's.

Theorem 1.3. *Let Γ be a graph with R red edges, G green edges, and B blue edges, and suppose that Γ has K properly colored K_4's. Then, $K \leq \frac{1}{4}(RGB)^{2/3}$. Furthermore, if $K = \frac{1}{4}(RGB)^{2/3} > 0$, then Γ is obtained from a balanced blowup of a properly colored K_4, by possibly adding a set of isolated vertices.*

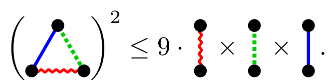
Note that we actually prove a slightly stronger result:

$$K \ \leq \ \frac{1}{4}\min\{RG, GB, RB\}. \tag{1}$$

We use similar methods to prove both theorems. The primary reason that similar methods work is that the extremal construction for both problems is the balanced blowup of a properly colored K_4. When using flag algebras, the obtained bounds are typically asymptotic. However, using blowups to move between finite graphs and asymptotics works remarkably well in this setting.

Note that flag algebras work with densities, instead of the number of edges. One difference is that the obtained results are not best possible when the density of the edges is more than $3/4$, as the blow-up of a K_4 is not sufficiently dense. In comparison, the extremal construction for maximizing the number of rainbow triangles in terms of the number of vertices in the problem of Erdős and Sós is a proper coloring of the complete graph obtained as an iterated blowup of a properly colored K_4 (see Figure 1(b)). The reason for this difference is that when the number of rainbow triangles is considered as a function of the number of vertices, then it is worthwhile to join every pair of vertices by an edge; on the other hand, when the number of rainbow triangles is considered as a function of the number of edges, it is not worthwhile to add additional edges inside the parts of the K_4 blowup.

Our proofs use the method of flag algebras, introduced by Razborov [17]. We recall the main definitions from flag algebras in the Appendix. Our main theoretical contribution is that our proofs are computer-free. Moreover, using the plain flag algebra method with sum-of-squares would not be computationally feasible. An equivalent reformulation of Theorem 1.1 in flag algebras that is approachable using semidefinite programming is


(2)

Verification of this inequality requires a computation using 3-edge-colored graphs on 6 vertices. There are 1,601,952 3-edge-colored graphs on 6 vertices up to isomorphism. Even after using symmetry between the colors, there are 268,835 configurations, and the resulting semidefinite program needs 540GB of RAM to numerically solve using CSDP solver. This would only be possible to achieve on high-memory nodes of a supercomputer. The next challenge would be rounding the numerical solution. Kiem, Pokutta and Spiegel [12] developed a rounding method that they used on rounding a numerical solution of a semidefinite program with 67,685 configurations. Their method may be usable in our case too. While performing the calculation may be doable with sufficient effort, our proof has the benefit of being easy to verify by the average reader without the need for a computer. Moreover, we managed to translate our proof to a counting proof by hand, which probably would not be possible in case of a large calculation.

Theorem 1.1 is motivated by the *joint* problem, which asks to determine the maximum number of joints in $\mathbb{R}^d$ determined by N lines, where a *joint* is a point P with a d-tuple of lines intersecting at P that spans the entire space $\mathbb{R}^d$. The joint problem was proposed in [7]. There, a construction of N lines with many joints was given: Choose k hyperplanes in $\mathbb{R}^d$ in general position. Then, the intersection of every $(d-1)$-tuple of the hyperplanes is a line, giving $N = \binom{k}{d-1}$ lines, and every d-tuple of planes provides a joint, generating $\binom{k}{d}$ joints. Guth [11, Section 2.5] conjectured that this construction is optimal, and Guth's conjecture was verified asymptotically by Yu and Zhao [19], and exactly by Chao and Yu [4].

Motivated by this construction, Yu and Zhao [19] defined *generically induced configurations* as follows: Let $\mathcal{H}$ be a set of hyperplanes in $\mathbb{R}^d$, and let L be a subset of the $(d-1)$-intersections of elements of $\mathcal{H}$. In this setting, every joint given by the line set L is an intersection point of d hyperplanes from $\mathcal{H}$. A generically induced configuration $(\mathcal{H}, L)$ has a natural representation as a $(d-1)$-uniform hypergraph, where each hyperplane of $\mathcal{H}$ corresponds to a vertex, and each line in L corresponds to a hyperedge. In the hypergraph representation, a joint corresponds to a complete $(d-1)$-uniform hypergraph on d vertices. In particular, in $\mathbb{R}^3$, a joint corresponds to a triangle in a graph.

The following *multijoint* problem was introduced by Zhang [20], who attributes the problem to Carbery: Let L_1, L_2, L_3 be three families of lines in $\mathbb{R}^3$; what is the maximum number of joints consisting of one line from each set L_i? When L_1, L_2, and L_3 are obtained by partitioning the line set of a generically induced configuration $(\mathcal{H}, L)$ into distinct color classes, this question asks for the maximum number of rainbow triangles in the graph representation of $(\mathcal{H}, L)$.

Given a 3-edge-colored graph corresponding to a generically induced configuration $(\mathcal{H}, L)$ whose line set is partitioned into three color classes L_1, L_2, L_3, a properly colored K_4 subgraph corresponds to a tetrahedron given by four planes of $\mathcal{H}$ in which opposite edges have the same color and each vertex is a multijoint. Therefore, Theorem 1.3 gives an exact solution for the maximum number of such properly colored tetrahedra in terms of $|L_1||L_2||L_3|$. While this problem is somewhat contrived, according to [19], a problem is "a rare instance in incidence geometry where the sharp constant is determined."

Note that the non-multicolor version of the tetrahedron problem also seems to be new, but solving it likely requires different methods.

Problem 1. *Given N lines in $\mathbb{R}^3$, what is the maximum number tetrahedra that they determine?*

The answer to Problem 1 likely comes from planes; that is, k planes generate $N = \binom{k}{2}$ lines, which generate $\binom{k}{4}$ tetrahedra.

Inspired by the rainbow triangle problem of Erdős and Sós [9], the second author poses the following question.

Question 1. *Does there exist a value $\varepsilon > 0$ and an infinite family $\mathcal{G}$ of 6-edge-colored graphs such that each $\Gamma \in \mathcal{G}$ has at least $(\frac{24}{215} + \varepsilon)\binom{|V(\Gamma)|}{4}$ rainbow copies of a K_4?*

The density $\frac{24}{215}$ of rainbow copies of K_4 can be obtained by one of two iterated blowups of a 6-edge-coloring of K_6 depicted in Figure 2. The second of these 6-edge-colorings is obtained by assigning each vertex of K_6 an element of $\mathbb{Z}_6$, and then assigning edge colors based on sums in $\mathbb{Z}_6$. In comparison, an iterated blowup of a properly colored K_4 gives density only $\frac{24}{252}$, which is slightly less.

We also wonder if there is a rainbow version of Theorem 1.3.

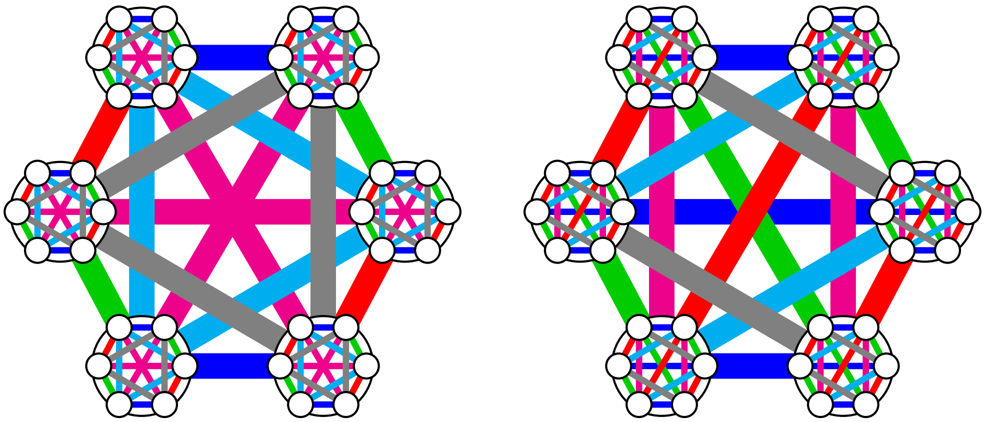


Figure 2: Two iterated constructions of a 6-edge-colored complete graph where rainbow K_4 has density 24/215.

Question 2. *Let Γ be a graph with edges colored by colors $C = \{1, \ldots, 6\}$. Denote by C_i the number of edges colored by color i. Let H be the number of rainbow copies of K_4 in Γ. Is it true that $H \leq \sqrt[3]{\prod_i C_i}$?*

Note that the bound from Question 2 would be tight for a blow-up of a fixed rainbow coloring of K_4, as well as graphs depicted in Figure 2, where the insides of the blow-ups are removed. We support the conjecture by proving a weaker version counting only one of the rainbow colorings of K_4.

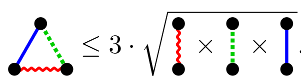
Theorem 1.4. *Let Γ be a graph with edges colored by colors $C = \{1, \ldots, 6\}$. Denote by C_i the number of edges colored by color i. Let H be the number of copies of K_4 in Γ with a fixed rainbow 6-edge-coloring. Then, $H \leq \sqrt[3]{\prod_i C_i}$.*

In Section 2 we give three proofs of Theorem 1.1, one uses flag algebras, one is a counting proof and the third uses entropy. In Section 3 we prove Theorem 1.3, and in Section 4 Theorem 1.4. We have some concluding remarks in Section 5 and a brief introduction to flag algebras in the Appendix.

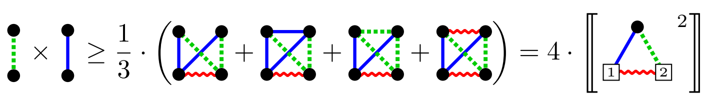
2 Proofs of Theorem 1.1 via flag algebras, counting and entropy

We assume the reader of this paper has some basic backgrounds in flag algebras; we provide some introduction to flag algebras in the Appendix, which should be sufficient to understand our proofs. First, we prove the asymptotic translation of Theorem 1.1 into the flag algebra setting.

Lemma 2.1.

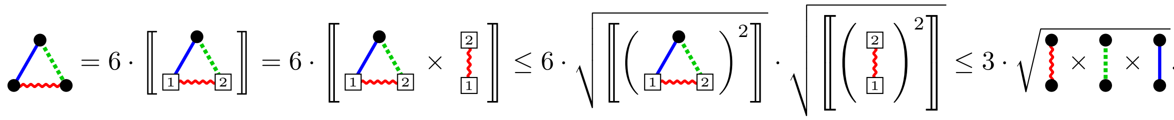


Proof. First, we observe the following flag algebra inequality


(3)

The inequality follows, as in each of the shown graphs on four vertices, there are six ways to partition the four vertices into parts A_1 and A_2 of size 2, and for exactly two of these partitions, A_1 induces a green edge while A_2 induces a blue edge. The identity holds, as the four 4-vertex graphs comprise all 4-vertex 3-edge-colored graphs in which a red edge uv belongs to two rainbow triangles, with u opposite the green edge and v opposite the blue edge in each triangle. Furthermore, given one of these four graphs H, the probability that a random injection $\{1,2\} \hookrightarrow V(H)$ both maps 1 to the unique vertex with red degree 1 and blue degree 2, and also maps 2 to the unique vertex with red degree 1 and green degree 2, is $\frac{1}{12}$.

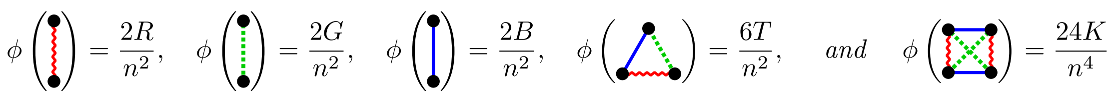
We also use Razborov's Cauchy-Schwarz inequality for flags [17, Theorem 3.14], which states that two flags F, G of a type σ satisfy $[\![F \times G]\!] \leq \sqrt{[\![F^2]\!][\![G^2]\!]}$. Combining this Cauchy-Schwarz inequality with the inequality (3), we obtain


(4)

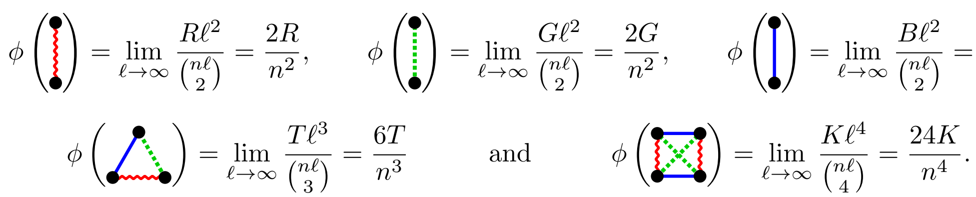
□

The inequality from Lemma 2.1 is valid for every limit of a convergent sequence of graphs. In order to obtain the inequality in Theorem 1.1 for a finite graph Γ, we create a convergent sequence of blowups of Γ with a limit $\phi \in Hom^+(\mathcal{A}, \mathbb{R})$, where $\mathcal{A}$ is the flag algebra $\mathcal{A}$ of unlabeled 3-edge-colored graphs (see Appendix A). The inequality from Lemma 2.1 applied to ϕ will give the desired inequality for Γ.

Lemma 2.2. *If Γ is a 3-edge-colored graph on n vertices with R red edges, G green edges, B blue edges, T rainbow triangles and K properly 3-edge-colored copies of K_4, then there exists a convergent sequence of 3-edge-colored graphs whose limit $\phi \in Hom^+(\mathcal{A}, \mathbb{R})$ satisfies*

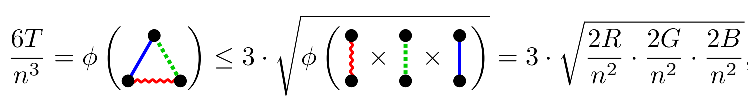


Proof. Let Γ be a 3-edge-colored graph on n vertices. Define the sequence $(\Gamma_\ell)_{\ell \geq 1}$, where Γ_ℓ is a graph obtained from Γ by replacing each of its vertices by an independent set of ℓ vertices and each edge of color c by a copy of $K_{\ell,\ell}$ with all edges of the $K_{\ell,\ell}$ colored c. This is a convergent sequence of graphs, and its limit corresponds to a positive homomorphism $\phi \in Hom^+(\mathcal{A}, \mathbb{R})$. Since each edge in Γ is replaced by ℓ^2 edges in Γ_ℓ of the same color, Γ_ℓ has $R\ell^2$, $G\ell^2$ and $B\ell^2$ red, green and blue edges respectively. Similarly, Γ_ℓ has $T\ell^3$ rainbow triangles and $K\ell^4$ properly 3-edge-colored copies of K_4. A straightforward calculation shows



□

Proof of Theorem 1.1. Let Γ be a 3-edge-colored graph on n vertices. Lemmas 2.1 and 2.2 imply


(5)

from which we conclude $T \leq \sqrt{2RGB}$. □

Now, we translate the flag algebras proof of Theorem 1.1 into a proof using only elementary counting arguments. First, we give a translation of the inequality (3).

Lemma 2.3. *Let Γ be a graph whose edges are colored red, green, and blue. Let S be the set of ordered vertex tuples $(u, v, x, y) \in V(\Gamma)^4$ for which uv is a red edge, ux and uy are blue edges, and vx and vy are green edges, as in Figure 3. Let S' be the set of pairs $(g, b) \in E(\Gamma)^2$ for which g is an unordered green edge and b is an unordered blue edge. Then, $|S| \leq |S'|$.*

Figure 3: Members of sets S and S' from Lemma 2.3.

Proof. To prove the lemma, we show that the function $f : S \to S'$ mapping $(u, v, x, y) \mapsto (\{u, x\}, \{v, y\})$ is injective. Indeed, choose an arbitrary element (g, b) in the image of f. We show that the vertices u, v, x, y can be uniquely determined from (g, b).

Case (i): Suppose that g and b share an endpoint z.
Then, the set $\{u, v, x, y\}$ contains at most three distinct vertices and induces three edges. Thus, u is the unique vertex incident to a red edge and a blue edge, v is the unique vertex incident to a red edge and a green edge, and $z = x = y$.

Case (ii): Suppose that g and b are vertex-disjoint.
Then, u is the endpoint of b with two blue neighbors and one red neighbor among the endpoints of g and b. Crucially, u is the unique vertex not having a green edge. Then, x is the endpoint of b distinct from u. Having identified u and x, v is the red neighbor of u, and y is the other blue neighbor of u.

As each element of S' in the image of f has at most one pre-image in S, we have $|S| \leq |S'|$. □

Now, we give a counting proof of Theorem 1.1 following (4).

Proof of Theorem 1.1. Let Γ be a graph whose edges are colored red, green, and blue, having R red, G green, and B blue edges. Let $\mathcal{R} \subseteq V(\Gamma)^2$ be the set of ordered pairs $uv \in V(\Gamma)^2$ that induce a red edge. Note that $|\mathcal{R}| = 2R$. For each $uv \in \mathcal{R}$, let $d^+(uv)$ denote the number of triples $uvw \in V(\Gamma)^3$ for which uw is blue and vw is green. Then, applying the Cauchy-Schwarz inequality, the number of rainbow triangles in Γ is

$$T = \sum_{uv \in \mathcal{R}} d^+(uv) \cdot 1 \leq \sqrt{\sum_{uv \in \mathcal{R}} d^+(uv)^2} \cdot \sqrt{2R}. \tag{6}$$

Observe that, given an ordered pair $uv \in \mathcal{R}$, $d^+(uv)^2$ counts the ordered pairs $xy \in V(\Gamma)^2$ for which ux, uy are blue edges and vx, vy are green edges. Therefore, $\sum_{uv \in \mathcal{R}} d^+(uv)^2$ is the number of ordered vertex tuples (u, v, x, y) for which uv is a red edge, ux and uy are blue edges, and vx and vy are green edges. Thus, by Lemma 2.3, $\sum_{uv \in \mathcal{R}} d^+(uv)^2 \leq G \cdot B$. Putting this together with (6), we conclude $T \leq \sqrt{2RGB}$. □

Using our counting proof of Theorem 1.1, we can in fact show uniqueness of the graph for which the upper bound on the number of rainbow triangles is attained. First, we need a lemma.

Lemma 2.4. *Let Γ be a graph whose edges are colored red, green, and blue. Assume that the following conditions hold for Γ:*

(a) For each pair e_1, e_2 of edges of distinct colors c_1 and c_2, there is an edge of the third color c_3 joining e_1 and e_2.

(b) There exists $d \geq 1$ such that every $v \in V(\Gamma)$ is incident to exactly d red, d green, and d blue edges. Then, Γ is a balanced blowup of a properly colored K_4.

Proof. First we show that Γ is connected. Indeed, suppose that $V(\Gamma)$ has two disjoint subsets V_1 and V_2 such that $\Gamma[V_1]$ and $\Gamma[V_2]$ are components of Γ. By (b), $\Gamma[V_1]$ contains a green edge g, and $\Gamma[V_2]$ contains a blue edge b. Then, by (a), a red edge joins an endpoint of g with an endpoint of b, a contradiction.

Now, fix a vertex $v \in V(\Gamma)$. Write $V_R \subseteq V(\Gamma)$ for the set of vertices of Γ that are joined to v by a red edge, and define V_G and V_B similarly. Define $V_0 \subseteq V(\Gamma)$ as the set of vertices in $V(\Gamma)$ that are not adjacent to v. As v has exactly d incident edges of each color, $|V_R| = |V_G| = |V_B| = d$. Furthermore, for each permutation (c_1, c_2, c_3) of the three symbols R, G, B, (a) implies that each $u \in V_{c_1}$ and each $u' \in V_{c_2}$ are joined by an edge of a color matching the symbol c_3.

Next, fix a vertex $w \in V_G$. As w is incident to exactly d edges of each color, (a) and (b) imply that the set of neighbors of w in red is exactly the set V_B, and the set of neighbors of w in blue is exactly the set V_R. Thus, as (a) also implies that every triangle is rainbow, V_R and V_B are independent sets.

Similarly, (a) and (b) imply that for each vertex $w' \in V_B$, the set of red neighbors of w' is exactly V_G; therefore, (a) implies that V_G is an independent set. Now, let U be the set of the d neighbors of w in green. As V_G is an independent set, $U \subseteq V_0$. Furthermore, by (a), for each vertex $x \in U$, x is a red neighbor of each vertex of V_R, and x is a blue neighbor of each vertex of V_B, and hence (a) also implies that x is a green neighbor of each vertex of V_G. Thus, using (b), we obtain that $N_\Gamma(v') \subseteq V_R \cup V_G \cup V_B \cup U$ for every vertex $v' \in V_R \cup V_G \cup V_B \cup U$. Since Γ is connected, it follows $U = V_0$.

Therefore, Γ is obtained from a properly colored K_4 by blowing up each vertex to one of the independent sets V_R, V_G, V_B, V_0, each of which contains exactly d vertices. □

Proof of Theorem 1.2. Let Γ be a graph whose edges are colored red, green, and blue, with exactly $T = \sqrt{2RGB} > 0$ rainbow triangles. We aim to show that Γ is obtained from a balanced blowup of a properly colored K_4, by possibly adding a set of isolated vertices. Write $V' \subseteq V(\Gamma)$ for the set of vertices in Γ incident to at least one edge. We also observe that each edge of Γ belongs to a rainbow triangle. Indeed, if e does not belong to a rainbow triangle, then e could be deleted without reducing the number of rainbow triangles. Assuming without loss of generality that e is red, $T = \sqrt{2RGB} > \sqrt{2(R-1)GB}$, contradicting Theorem 1.1.

As Theorem 1.1 states that $T \leq \sqrt{2RGB}$, the fact that $T = \sqrt{2RGB}$ implies that each inequality in the proof of Theorem 1.1 is an equality. We show that if each inequality in the proof of Theorem 1.1 holds, then conditions (a) and (b) of Lemma 2.4 hold for $\Gamma[V']$, implying that Γ is obtained from a balanced blowup of a properly colored K_4 by adding a set of isolated vertices.

First, we analyze the inequality given by Lemma 2.3. If $|S| < |S'|$ in the lemma, then the inequality $\sum_{uv \in \mathcal{R}} d^+(uv)^2 \leq G \cdot B$ in Theorem 1.1 is strict, a contradiction; therefore, $|S| = |S'|$. Hence, for every green edge g of G and every blue edge b of G, the endpoints of g and b form a set $\{u, v, x, y\}$ for which uv is a red edge, ux and uy are blue edges, and vx and vy are green edges. In particular, a red edge joins an endpoint of g with an endpoint of b. By repeating the argument with permuted colors, we observe that the condition (a) of Lemma 2.4 holds for $\Gamma[V']$.

Next, we claim that condition (b) of Lemma 2.4 holds for $\Gamma[V']$.

Claim 2.5. *There is a $d \geq 1$ such that every $v \in V'$ is incident to exactly d red, d green, and d blue edges.*

Proof. We analyze the Cauchy-Schwarz inequality $\sum_{uv \in \mathcal{R}} d^+(uv) \cdot 1 \leq \sqrt{\sum_{uv \in \mathcal{R}} d^+(uv)^2} \cdot \sqrt{2R}$ in the proof of Theorem 1.1. The Cauchy-Schwarz inequality is an equality if and only if the values $d^+(uv)$ are equal for all pairs $uv \in \mathcal{R}$. By repeating the argument with permuted colors, and recalling that each edge of Γ belongs to at least one rainbow triangle, we conclude that there exists a $d \geq 1$ such that the following holds for every permutation (c_1, c_2, c_3) of the three colors:

($\star$) If $u, v \in V(\Gamma)$ induce an edge of color c_1, then there are exactly d vertices w for which u, w induce an edge of color c_2 and v, w induce an edge of color c_3.

Now, consider a vertex $u \in V'$. As $u \in V'$, u has a neighbor u', and without loss of generality, uu' is red. Then, by ($\star$), u belongs to a rainbow triangle in which u is incident to a green edge, as well as a rainbow triangle in which u is incident to a blue edge. Therefore, u is incident to at least one edge of each color.

Next, let (c_1, c_2, c_3) be an arbitrary permutation of the three colors, and let v be a vertex for which uv has color c_1. By ($\star$), there exist at least d vertices w for which uw has color c_2. Furthermore, for each vertex $w \in V(\Gamma)$ for which uw has color c_2, condition (a) of Lemma 2.4 implies that vw has color c_3. Therefore, $V(\Gamma)$ has exactly d vertices w for which uw has color c_2. Since u and c_2 were chosen arbitrarily, u has exactly d incident edges of each color. □

Since conditions (a) and (b) of Lemma 2.4 hold, $\Gamma[V']$ is a balanced blowup of a properly colored K_4. Therefore, Γ is obtained from a balanced blowup of a properly colored K_4 by possibly adding a set of isolated vertices. $\square$

Finally, we present an entropy version of our proof of Theorem 1.1, which is inspired by [6]. We assume familiarity with the entropy of a discrete random variable X, defined as $H(X) = -\sum_x \mathbb{P}(X = x)\log_2(\mathbb{P}(X = x))$, as well as some related results.

Proof of Theorem 1.1. Let $v_g v_b v_r \in V(\Gamma)^3$ be a rainbow triangle sampled uniformly at random from Γ, where $v_g v_b$, $v_b v_r$ and $v_r v_g$ are red, green and blue edges respectively. We resample a vertex v'_r so that $v_g v'_r$ and $v_b v'_r$ are blue and green edges, and v_r and v'_r are conditionally independent given v_g, v_b and $H(v'_r|v_g, v_b) = H(v_r|v_g, v_b)$.

By the chain rule, we have $H(v_g, v_b, v_r) = H(v_r|v_g, v_b) + H(v_g, v_b)$ and using $H(v'_r|v_g, v_b) = H(v_r|v_g, v_b)$, it follows $H(v_g, v_b, v_r) = H(v'_r|v_g, v_b) + H(v_g, v_b)$. By adding these two equations and using conditional independence, we obtain

$$\begin{aligned} 2H(v_g, v_b, v_r) &= H(v_r|v_g, v_b) + H(v_g, v_b) + H(v'_r|v_g, v_b) + H(v_g, v_b) = H(v_r, v'_r|v_g, v_b) + 2H(v_g, v_b) \\ &= H(v_g, v_b, v_r, v'_r) + H(v_g, v_b). \end{aligned}$$

Since (v_g, v_b) is an ordered red edge, it follows $H(v_g, v_b) \le \log_2(2R)$. Recall that S is the set of ordered vertex tuples $(u, v, x, y) \in V(\Gamma)^4$ for which uv is a red edge, ux and uy are blue edges, and vx and vy are green edges, see Figure 3. Notice that $(v_g, v_b, v_r, v'_r) \in S$, therefore $H(v_g, v_b, v_r, v'_r) \le \log_2(|S|)$. By Lemma 2.3 we have $H(v_g, v_b, v_r, v'_r) \le \log_2(|S'|) = \log_2(G) + \log_2(B)$. We conclude

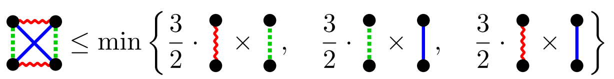
$$T = 2^{H(v_g, v_b, v_r)} \le 2^{\frac{1}{2}H(v_g, v_b, v_r, v'_r) + \frac{1}{2}H(v_g, v_b)} \le \sqrt{2RGB}.$$

$\square$

3 Proof of Theorem 1.3

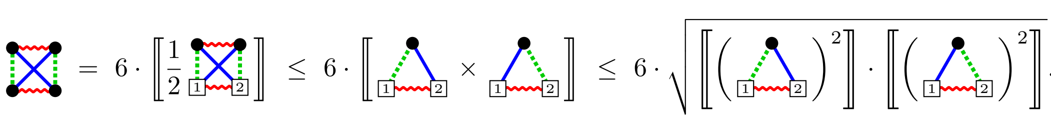
Consider a graph Γ with R red edges, G green edges, and B blue edges, and let K be the number of properly colored K_4 subgraphs of Γ. First, we show a short proof of the stronger inequality $K \le \frac{1}{4} \cdot \min\{RG, GB, RB\}$ using the language of flag algebras.

Lemma 3.1.



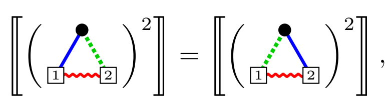
(7)

Proof. First, by the Cauchy-Schwarz inequality,

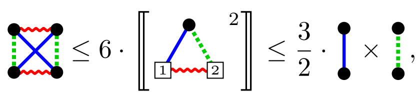


(8)

Since



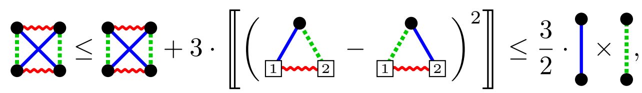
we obtain



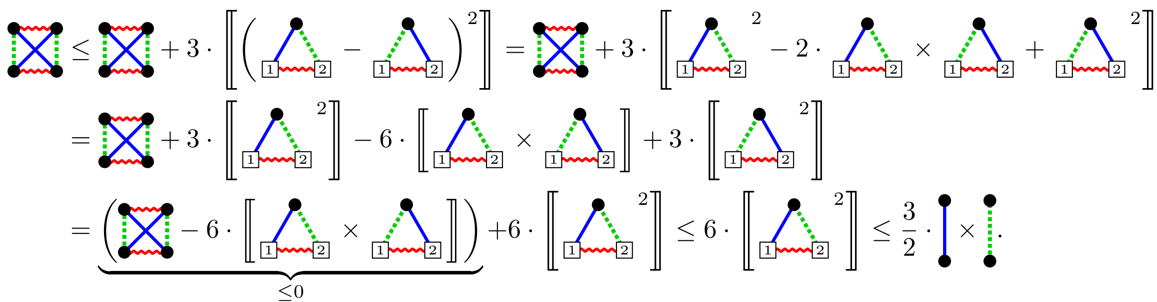
with the last inequality following from (3). By repeating the argument with a rooted green edge and a rooted blue edge,

The above three inequalities imply (7). $\square$

We can strengthen Lemma 3.1 by adding a square, which is typical for applications of flag algebras:



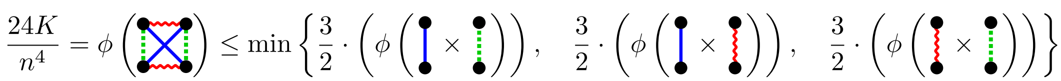
where the last inequality can be checked using the equations and inequalities in the proof of Lemma 3.1:



Lemma 3.1 can be translated to an exact inequality via blowups in the same way as in Lemma 2.1.

Claim 3.2. $K \leq \frac{1}{4} \cdot \min\{RG, GB, RB\}$.

Proof. Let Γ be a 3-edge-colored graph on n vertices. Lemma 2.2 and (7) imply



$$\leq \min\left\{\frac{3}{2}\cdot\left(\frac{2R}{n^2}\cdot\frac{2G}{n^2}\right),\quad \frac{3}{2}\cdot\left(\frac{2G}{n^2}\cdot\frac{2B}{n^2}\right),\quad \frac{3}{2}\cdot\left(\frac{2B}{n^2}\cdot\frac{2R}{n^2}\right)\right\},$$

yielding the statement of the claim. $\square$

The following is an alternative proof of Claim 3.2 that does not use flag algebras.

Proof of Claim 3.2. Let K be the number of properly colored K_4 subgraphs of Γ. Let $\mathcal{R}$ be the set of ordered vertex pairs $uv \in V(\Gamma)^2$ for which uv induces a red edge. For each $uv \in \mathcal{R}$, let $d_K(uv)$ be the number of ordered pairs $wx \in \mathcal{R}$ for which uw and vx are green, and ux and vw are blue. Finally, for each $uv \in \mathcal{R}$, define $d^+(uv)$ as the number of triples $uvw \in V(G)^3$ for which uw is blue and vw is green, and define $d^-(uv)$ as the number of triples $uvw \in V(G)^3$ for which uw is green and vw is blue. Observe that for each $uv \in \mathcal{R}$,

$$d_K(uv) \leq d^-(uv)\cdot d^+(uv) \qquad \text{and} \qquad \sum_{uv\in\mathcal{R}} d^-(uv)^2 = \sum_{uv\in\mathcal{R}} d^+(uv)^2. \tag{9}$$

Thus, by using the Cauchy-Schwarz inequality

$$4K = \sum_{uv\in\mathcal{R}} d_K(uv) \leq \sum_{uv\in\mathcal{R}} d^-(uv)\cdot d^+(uv) \leq \sqrt{\left(\sum_{uv\in\mathcal{R}} d^-(uv)^2\right)\cdot\left(\sum_{uv\in\mathcal{R}} d^+(uv)^2\right)} = \sum_{uv\in\mathcal{R}} d^+(uv)^2. \tag{10}$$

As observed in the proof of Theorem 1.1, $\sum_{uv\in\mathcal{R}} d^+(uv)^2$ is the number of ordered vertex tuples (u,v,x,y) for which uv is a red edge, ux and uy are blue edges, and vx and vy are green edges. By Lemma 2.3, $\sum_{uv\in\mathcal{R}} d^+(uv)^2 \le G\cdot B$. Putting this together with (10), we obtain $4K \le GB$.

By symmetry, we also have $4K \le RB$ and $4K \le RG$. Therefore, these three upper bounds imply

$$4K \le \min\{RG,\ GB,\ RB\}.$$

□

Proof of Theorem 1.3. By taking the geometric mean of the upper bounds in Claim 3.2, it follows that $K \le \frac{1}{4}(RGB)^{2/3}$. To complete the proof of Theorem 1.3, we show that if $K = \frac{1}{4}(RGB)^{2/3} > 0$, then Γ is obtained from a balanced blowup of a properly colored K_4 by adding a set of isolated vertices.

If one the expressions $4K \le R\cdot G$, $4K \le R\cdot B$ and $4K \le B\cdot G$ is a strict inequality, then it follows $(4K)^3 < (R\cdot G\cdot B)^2$. Therefore, if $K = \frac{1}{4}(R\cdot G\cdot B)^{2/3}$, then the inequality in Lemma 2.3, as well as the inequalities obtained from it by permuting colors, are tight.

We again write $V' \subseteq V(\Gamma)$ for the set of vertices with at least one incident edge. As in the proof of Theorem 1.2, we assume that each edge of Γ belongs to at least one properly colored K_4.

Now, as in the proof of Theorem 1.2, tightness of Lemma 2.3 implies condition (a) of Lemma 2.4. Furthermore, for each pair $u,v \in V(\Gamma')$, the fact that each of u and v belongs to a rainbow K_4 implies that u is incident to a green edge and v is incident to a blue edge; thus, condition (a) of Lemma 2.4 implies that a path joins u and v and hence that $\Gamma[V']$ is connected. Furthermore, if $K = \frac{1}{4}(RGB)^{2/3}$, then the Cauchy-Schwarz inequality in (10) is tight, implying that for each $uv \in \mathcal{R}$, $d^+(uv) = d^-(uv)$.

Furthermore, for each $uv \in \mathcal{R}$, condition (a) of Lemma 2.4 implies that $d_G(u) = d^-(uv) = d^+(uv) = d_B(u)$, where $d_G(u)$ and $d_B(u)$ denote the number of incident green edges and incident blue edges to u, respectively. Additionally, $d_G(u) = d^-(uv) = d_B(v)$. Therefore, by permuting colors and applying the argument to each adjacent pair $uv \in \Gamma[V']$, and using the fact that $\Gamma[V']$ is connected, condition (b) of Lemma 2.4 holds for $\Gamma[V']$. Thus, Lemma 2.4 implies that $\Gamma[V']$ is a balanced blowup of a properly colored K_4, completing the proof of Theorem 1.3. □

To conclude this section, we provide an entropy version of our proof of Claim 3.2.

Proof of Claim 3.2. Let $(v_1,v_2,v_3,v_4) \in V(\Gamma)^4$ be a tuple of vertices sampled uniformly at random from the tuples such that the edges v_1v_2, v_3v_4 are red, v_1v_3, v_2v_4 are blue and v_1v_4, v_2v_3 are green, see Figure 4. From the definition of (v_1,v_2,v_3,v_4), it follows $H(v_1,v_2,v_3,v_4) = \log_2(4K)$. By the chain rule we have

$$\begin{aligned} H(v_1,v_2,v_3,v_4) &= H(v_1|v_2,v_3,v_4) + H(v_2,v_3,v_4) = H(v_1|v_2,v_3,v_4) + H(v_2|v_3,v_4) + H(v_3,v_4) \\ &\le H(v_1|v_3,v_4) + H(v_2|v_3,v_4) + H(v_3,v_4), \end{aligned} \tag{11}$$

where the inequality follows from dropping the conditioning on v_2. We resample a vertex v_1' so that the edges $v_1'v_3$ and $v_1'v_4$ are blue and green, respectively, and the vertices v_1 and v_1' are conditionally independent given v_3,v_4 and $H(v_1'|v_3,v_4) = H(v_1|v_3,v_4)$. Similarly, we resample a vertex v_2' so that the edges $v_2'v_4$ and $v_2'v_3$ are blue and green, respectively, and the vertices v_2 and v_2' are conditionally independent given v_3,v_4 and $H(v_2|v_3,v_4) = H(v_2'|v_3,v_4)$. From (11) it follows that

$$\begin{aligned} H(v_1,v_2,v_3,v_4) &\le \frac{1}{2}\left(H(v_1|v_3,v_4) + H(v_2|v_3,v_4) + H(v_3,v_4) + H(v_1'|v_3,v_4) + H(v_2'|v_3,v_4) + H(v_3,v_4)\right) \\ &= \frac{1}{2}\left(H(v_1,v_1'|v_3,v_4) + H(v_2,v_2'|v_3,v_4) + 2H(v_3,v_4)\right) = \frac{1}{2}\left(H(v_3,v_4,v_1,v_1') + H(v_3,v_4,v_2,v_2')\right). \end{aligned}$$

Notice that $(v_3,v_4,v_1,v_1'), (v_4,v_3,v_2,v_2') \in S$; therefore $H(v_3,v_4,v_1,v_1'), H(v_3,v_4,v_2,v_2') \le \log_2(|S|)$. By Lemma 2.3 we have $\log_2(|S|) \le \log_2(G) + \log_2(B)$. We conclude

$$4K = 2^{H(v_1,v_2,v_3,v_4)} \le 2^{\frac{1}{2}\left(H(v_3,v_4,v_1,v_1') + H(v_3,v_4,v_2,v_2')\right)} \le G\cdot B.$$

□

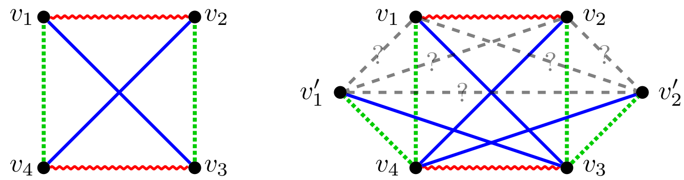
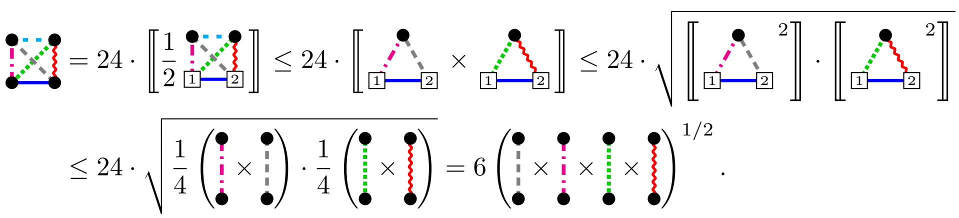


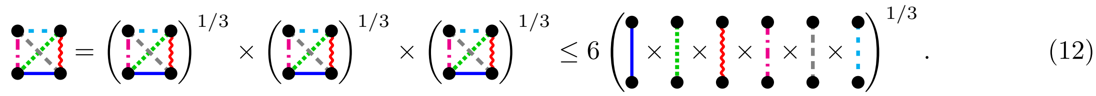
Figure 4: A drawing of $\Gamma[v_1, v_2, v_3, v_4]$ and the graph obtained by adding the resampled vertices v_1', v_2'.

4 Proof of Theorem 1.4

Proof of Theorem 1.4. An application of the Cauchy-Schwarz inequality and a 6-color analogue of (3) give



By symmetry we get



The analog of Lemma 2.2 holds also for edges in the 6-edge-colored setting and for rainbow copies of K_4. Let Γ be a graph and ϕ the limit of convergent sequence of balanced blow-ups of Γ. If Γ has C_i edges of color i for $i \in \{1, \ldots, 6\}$ and H copies of a fixed rainbow coloring of K_4, we obtain

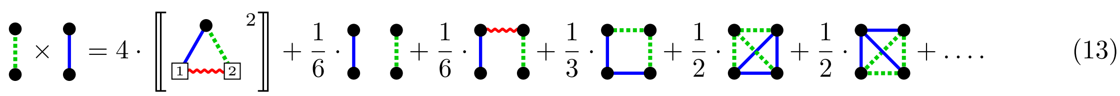
where c_i is an edge of color i. Combining this with (12), we get

$$\frac{24H}{n^4} \leq 6 \left(\prod_{i=1}^{6} \frac{2C_i}{n^2} \right)^{1/3},$$

which simplifies to $H \leq \prod_{i=1}^{6} C_i$. □

5 Concluding Remarks

We note that an argument that is similar to but more tedious than the proof of Theorem 1.2 shows that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that when $T^2 \geq (\sqrt{2} - \delta)\sqrt{RGB}$, then Γ can be transformed into a balanced blowup of a properly colored K_4 by changing at most $\varepsilon |V(\Gamma)|^2$ vertex pairs. A precise version of (3) states that



Thus, if $T^2 \geq (\sqrt{2} - \delta)\sqrt{RGB}$, then the unlabeled graphs on the right-hand side of (13) have density at most, say, 100δ. Then, when δ is sufficiently small, an induced removal lemma (see for example [1]) implies that Γ can be transformed into an edge-colored graph Γ' that contains none of the unlabeled graphs on the right-hand side of (13), or a color permutation thereof, by changing at most $\frac{1}{2}\varepsilon|V(\Gamma)|^2$ vertex pairs. These forbidden induced subgraphs imply that Γ' is obtained from a blowup of a properly colored K_4 by adding isolated vertices. Furthermore, when δ is sufficiently small, the bound $T^2 \geq (\sqrt{2}-\delta)\sqrt{RGB}$ implies that Γ' is almost balanced and hence it can be made balanced by editing at most another $\frac{1}{2}\varepsilon|V(\Gamma)|^2$ edges. We decided not to present the proof, because it is tedious and not particularly enlightening. The proof of Theorem 1.2 uses condition (a) of Lemma 2.4, which is obtained from lower-order terms and cannot be obtained in a straight-forward way from the flag algebra calculation.

6 Acknowledgments

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A Flag Algebras

The purpose of this section is to introduce the main definitions for flag algebras used in this paper, however, we are not attempting to give a complete introduction to flag algebras. It should allow the reader to follow the calculations and make the paper self-contained. An interested reader may look at [3, 8, 10, 13, 14, 16].

To simplify the notation, denote the number of vertices of a graph G by $v(G)$. Denote all graphs on n vertices up to isomorphism by $\mathcal{F}_n$ and the union of them by $\mathcal{F}$. The *density* of a graph G in a graph H is

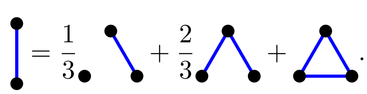
$$p(G,H) = \frac{|\{X : X \subseteq V(H), H[X] \cong G\}|}{\binom{v(H)}{v(G)}}.$$

A sequence of graphs $(G_n)_{n\geq 1}$ of increasing orders is *convergent* if for every $H \in \mathcal{F}$, the density of H in $(G_n)_{n\geq 1}$ converges, i.e., $\lim_{n\to\infty} p(H, G_n)$ exists. Examples of convergent sequences are a sequence of balanced blowups of increasing size a fixed graph or a sequence of Erdős-Rényi random graphs $G_{n,p}$, where p is a constant and n tends to infinity. By compactness, every sequence of graphs has a convergent subsequence [17, Theorem 3.2]. Denote by $\phi = \phi(H)$ the limits of $p(H, G_n)$, which is a function $\mathcal{F} \to [0, 1]$. Razborov showed [17, Theorem 3.3] that ϕ is a homomorphism from a certain algebra $\mathcal{A}$ to $\mathbb{R}$. Positive homomorphims from $\mathcal{A}$ to $\mathbb{R}$, denoted by $Hom^+(\mathcal{A}, \mathbb{R})$, are homomorphisms ϕ, where $\mathcal{F} \to [0, 1]$. All these homomorphisms are limits of convergent sequences [17, Theorem 3.3].

We use graphs for accessibility of the explanation. The graphs can be replaced by k-uniform hypergraphs, permutations, or other models. In general one needs to have vertices with relations of finite arity. Different models result in different algebras $\mathcal{A}$.

If an expression using flags is valid for all $\phi \in Hom^+(\mathcal{A}, \mathbb{R})$, then we omit writing ϕ to decrease clutter in the notation.

The algebra $\mathcal{A}$ is obtained from formal finite linear combinations of graphs in $\mathcal{F}$ by factoring $\mathcal{K} := \mathrm{span}(\{F - \sum_{F' \in \mathcal{F}_{v(F)+1}} p(F, F')F) : \forall F \in \mathcal{F}\}$. The expressions in $\mathcal{K}$ enforce in $\mathcal{A}$ identities such as

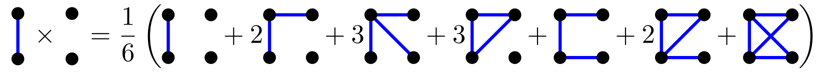


The intuitive idea is that calculations happen with linear combinations of densities of small graphs in a very large graph. The product of $F_1, F_2 \in \mathcal{F}$ is defined as

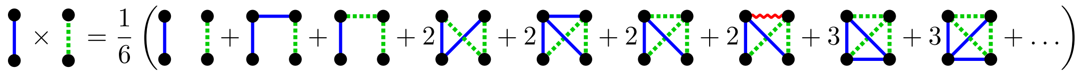
$$F_1 \times F_2 = \sum_{F \in \mathcal{F}_{v(F_1)+v(F_2)}} p(F_1, F_2; F) \cdot F, \tag{14}$$

where $p(F_1, F_2; F)$ is the probability that $F[X] \cong F_1$ and $F[V(F) \setminus X] \cong F_2$ for $X \subseteq V(F)$ with $|X| = v(F_1)$ which is chosen uniformly at random. One could think of the left hand-side of (14) as choosing uniformly independently at random in a large graph sets X_1 of $v(F_1)$ vertices and X_2 of $v(F_2)$ vertices and asking if

X_1 induces a copy of F_1 and X_2 induces a copy of F_2. If the underlying graph is very large, then X_1 and X_2 are typically disjoint. The right hand-side of (14) lists the options for $X_1 \cup X_2$. For $\mathcal{F}$ of simple graphs



For $\mathcal{A}$ of 3-edge colored graphs

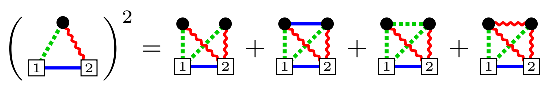


Extending (14) linearly defines the product on $\mathcal{A}$.

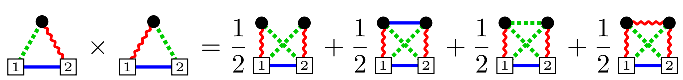
Extremal graph theory arguments often include counting over a fixed substructure. Some examples for such counting are $\sum_{v \in V(G)} d(v)$ or $\sum_{uv \in E(G)} |N(u) \cap N(v)|$, where G is a graph. The entries in these sums have some fixed distinguished vertices. In flag algebras, this is modeled using graphs with $\ell \geq 0$ vertices labeled by $\{1, 2, \ldots, \ell\}$. The graph on ℓ vertices, where each of the ℓ vertices are labeled is called a *type*. A type is usually denoted by σ and the resulting algebra by $\mathcal{A}^\sigma$. The main new properties of them are that the isomorphisms must preserve the labeling of the labeled vertices and in the definition of product, the labeled vertices are shared. More formally, let (F_1, θ_1) and (F_2, θ_2) be two labeled flags of the same type σ, where F_i is an unlabeled graph and $\theta_i : [\ell] \hookrightarrow V[F_i]$ is an injective map indicating the labeled vertices for $i \in \{1, 2\}$. Recall that (F_i, θ_i) being of type σ means that $(F_i[\mathrm{Im}(\theta_i)], \theta_i)$ is isomorphic to σ. The product is defined as

$$(F_1, \theta_1) \times (F_2, \theta_2) = \sum_{(F,\theta) \in \mathcal{F}^\sigma_{v(F_1)+v(F_2)-\ell}} p((F_1, \theta_1), (F_2, \theta_2); (F, \theta)) \cdot (F, \theta),$$

where $p((F_1, \theta_1), (F_2, \theta_2); (F, \theta))$ is the probability that the set $X \subseteq V(F) \setminus \mathrm{Im}(\theta)$ with $|X| = v(F_1) - \ell$ sampled uniformly at random satisfies that $(F[X \cup \mathrm{Im}(\theta)], \theta)$ is isomorphic to (F_1, θ_1) and $(F[V(F) \setminus X], \theta)$ is isomorphic to (F_2, θ_2). In 3-edge colored graphs



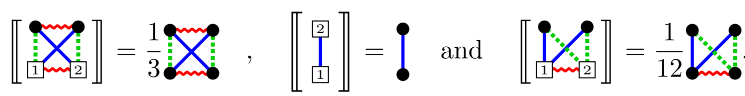
and



Expressions $\frac{1}{v} \sum_{v \in V(G)} d(v)$ or $\frac{1}{|E(G)|} \sum_{uv \in E(G)} |N(u) \cap N(v)|$ for a graph G are analogous to a linear unlabeling operator in flag algebras $[\![\cdot]\!]$. Let (F, θ) be a labeled graph, where F is an unlabeled graph and an injective map $\theta : [\ell] \to V(F)$ gives the labeling. Then,

$$[\![(F, \theta)]\!] = c_F F,$$

, where c_F is the probability that $(F, \theta) \cong (F, \theta')$ where $\theta' : [\ell] \to V(F)$ is an injective map chosen uniformly at random. In other words, randomly label ℓ vertices of F and ask if the original labeled graph is obtained. For example



Notice that the multiplication is defined only for graphs, where the subgraphs induced by the labeled vertices, i.e., the types, are the same.

With these definitions, there is an analogue of the Cauchy-Schwarz inequality [17, Theorem 3.14] that states for $f, g \in \mathcal{A}^\sigma$

$$[\![f^2]\!] \cdot [\![g^2]\!] \geq [\![fg]\!]^2.$$

All calculations presented in this paper should formally be surrounded by $\phi(\cdots)$ that is a limit of a convergent sequence of graphs. The calculations would be on graph densities. Since wrapping the calculations in $\phi(\cdot)$ adds notation but it is not useful for the calculation itself, they are usually omitted. The calculations are valid for any choice of ϕ and can be intuitively thought of as calculations in large graphs. Notice that the results from flag algebras are valid only in limits. When interpreting the calculations in large graphs, they come with $o(1)$ additive error. In particular, the calculations do not directly imply any meaningful thing if the convergent sequence consists of sparse graphs. In the flag algebras world, a sequence of sparse graphs is indistinguishable from graphs with no edges at all.