Weighted Turán theorems with applications to Ramsey-Turán type of problems

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Abstract

We study extensions of Turán Theorem in edge-weighted settings. A particular case of interest is when constraints on the weight of an edge come from the order of the largest clique containing it. These problems are motivated by Ramsey-Turán type problems. Some of our proofs are based on the method of graph Lagrangians, while the other proofs use flag algebras. Using these results, we prove several new upper bounds on the Ramsey-Turán density of cliques. Other applications of our results are in a recent paper of Balogh, Chen, McCourt and Murley.

Keywords: Ramsey-Turán, weighted Turán, weighted graphs, flag algebras.

1 Introduction

1.1 Ramsey-Turán theory

The maximum number of edges in a K_k-free graph on n vertices is achieved by the balanced complete $(k-1)$-partite graph, which is known as the *Turán Graph $T(n, k-1)$*. Observe that $T(n, k-1)$ has an independent set of size at least $n/(k-1)$. On the other hand, Ramsey's theorem asserts that for every k and ℓ there exists a finite number $R(k, \ell)$ such that every graph with $R(k, \ell)$ vertices contains either a clique of order k or an independent set of order ℓ. The combination of these two theorems motivated Sós [12] to introduce the so-called Ramsey-Turán theory. The typical problem asks to determine the maximum number of edges in a K_{k+1}-free n-vertex graph with independence number at most ℓ. By Ramsey's theorem, such a graph exists only if $n < R(k+1, \ell+1)$. The *inverse Ramsey number $Q(k, n) = \ell$* is the smallest possible independence number in a K_k-free graph on n vertices, i.e., ℓ is the maximum integer such that $R(k, \ell) \leq n$. When investigating the asymptotic behavior of the maximum number of edges in K_{k+1}-free n-vertex graphs with independence number at most ℓ, usually n goes to infinity, k is a fixed constant, and ℓ depends on n such that $\ell \geq Q(k, n)$.

More generally, we may require that every large subset of vertices spans not only an edge, but a clique of a given size, leading to the following definition. The *p-independence number* of a graph G is

$$\alpha_p(G) := \max \left\{ |U| : U \subseteq V(G) \text{ and } G[U] \text{ is } K_p\text{-free} \right\}.$$

Notice that $\alpha_2(G)$ is the independence number of G. The *Ramsey-Turán number $RT_p(n, K_q, m)$* is the maximum number of edges in an n-vertex K_q-free graph with $\alpha_p(G) \leq m$. The *Ramsey-Turán density* (in case it exists, which is expected) is

$$\varrho_p(q) := \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{RT_p(n, K_q, \varepsilon n)}{\binom{n}{2}}.$$
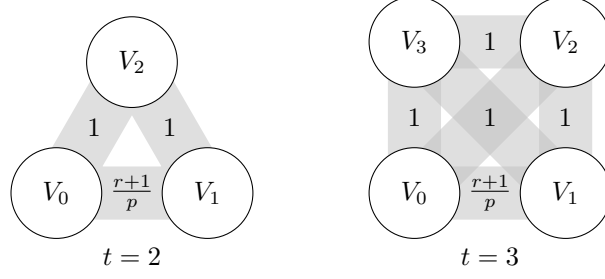
Figure 1: Construction from Conjecture 1.1 for $t = 2$ and $t = 3$. Recall $q = pt + r + 2$.

These types of problems were first studied by Sós [12], and Erdős and Sós [6] who determined the value of $\varrho_2(q)$ for all odd q. Since their work, determining Ramsey-Turán densities for various p and q has become a classical problem in extremal combinatorics.

The case when $p = 2$ and q is even has proved to be significantly harder. First non-trivial upper bound was proven by Szemerédi [13] using his celebrated regularity lemma. In terms of lower bounds, a big breakthrough was by Bollobás and Erdős [3], who constructed n-vertex K_4-free graphs with $\frac{n^2}{8} - o(n^2)$ edges and independence number $o(n)$. Building on these works, the case $p = 2$ was fully settled by Erdős, Hajnal, Simonovits, Sós and Szemerédi [4]:

$$\varrho_2(2t + 1) = \frac{t - 1}{t} \qquad \text{for all } t \geq 1, \qquad \text{and} \qquad \varrho_2(2t) = \frac{3t - 5}{3t - 2} \qquad \text{for all } t \geq 2.$$

Furthermore, in [4], the authors conjectured that the asymptotically extremal graphs for $\varrho_p(q)$ exhibit a certain periodic structure.

Conjecture 1.1 (Conjecture [4]). *The asymptotically extremal graph G for $\varrho_p(q)$ has the following structure. Let $q = pt + r + 2$, where $t \in \mathbb{N}$ and $r \in \mathbb{Z}_p$. Then, there exists a partition $V(G) = V_0 \cup V_1 \cup \cdots \cup V_t$ such that*

- $e(G[V_i]) = o(n^2)$ *for all* $0 \leq i \leq t$;

- $d_G(V_0, V_1) = \frac{r+1}{p} - o(1)$, *and degrees in* $G[V_0, V_1]$ *differ by* $o(n)$;

- $d_G(V_i, V_j) = 1 - o(1)$ *for all pairs* $\{i, j\} \neq \{0, 1\}$.

In particular,

$$\varrho_p(q) = \varrho_p^\star(q) := \frac{(t - 1)(2p - r - 1) + r + 1}{t(2p - r - 1) + r + 1}.$$

Non-trivial lower bounds are rare for $p \geq 3$, a remarkable result is due to Balogh and Lenz [1] who used a product construction utilizing the so-called Bollobás-Erdős graph mentioned above. A recent breakthrough was by Liu, Reiher, Sharifzadeh, and Staden [8], using complex high dimensional spheres. In particular, one of their constructions implies that $\varrho_3(5) = 1/6$. Furthermore, using a different construction, they disproved Conjecture 1.1 by showing $\varrho_{16}(22) = 1/6 > 5/32 = \varrho_{16}^\star(22)$. More importantly, their result implies that the structure of the extremal graphs is much more intricate than previously conjectured.

Our main contribution to Ramsey-Turán theory is proving new upper bounds as given by the following theorem.

Theorem 1.2. *The following bounds hold:*

$$\varrho_4(11) \leq \frac{4}{7}, \qquad \varrho_5(12) \leq \frac{10}{19}, \qquad \varrho_6(12) \leq \frac{5}{12}, \qquad \varrho_6(14) \leq \frac{12}{23}.$$

This implies $\varrho_5(12) = \frac{10}{19}$.

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To prove Theorem 1.2 we convert the problem of determining upper bounds for $\varrho_p(q)$ into an edge-weighted Turán setting, as outlined below, and then use the method of flag algebras introduced by Razborov [11].

Let us now briefly explain how the upper bounds for $\varrho_p(q)$ are obtained. First, one applies Szemerédi's regularity lemma to obtain an ε-regular equipartition of a given graph G. This gives an edge-weighted cluster graph, where the weights come from the densities of the corresponding regular pairs. Recall that the clusters in the partition have $\Omega(n)$ vertices. By the assumption on $\alpha_p(G)$, every linear subset of a cluster contains a copy of K_p. This allows one to show that certain weighted configurations in the cluster graph lead to the existence of a K_q in G. For example, to prove that $\varrho_2(4) \leq 1/4$, one shows how to embed a K_4 in G if there is a triangle in the cluster graph with all three pairwise densities bounded away from 0 as well as how to embed a K_4 in G if there are two clusters with density at least roughly $1/2$. After finding such weighted configurations which lead to the existence of the desired clique, the task reduces to a weighted Turán problem which we discuss next. The main obstacle for obtaining further upper bounds on Ramsey-Turán densities has been a lack of such weighted Turán results.

1.2 Weighted Turán problems

Given a graph G, an *edge weighting* w is a mapping $w : E(G) \to [0, 1]$. We extend w to G as a scaled sum of the weights of all its edges

$$w(G) := \frac{2}{n^2} \sum_{e \in E(G)} w(e).$$

Note that if we consider non-edges as edges of weight 0 then $w(G)$ is asymptotically the average weight of edges of G, which is $w(G) \cdot \frac{n}{n-1}$. We scale $w(G)$ by $2/n^2$ rather than $\binom{n}{2}^{-1}$ since this gives nicer extremal values.

Given a positive integer r, a *weighted clique* is a pair (r, f) where f is a function $f : \binom{[r]}{2} \to [0, 1]$. Given a graph G with an edge weighting w, we say that G contains a weighted clique (r, f) if there is an injective function $\phi : [r] \to V(G)$ such that for all $1 \leq i < j \leq r$, $\phi(i)\phi(j) \in E(G)$ and $w(\phi(i)\phi(j)) > f(ij)$. Given a set $\mathcal{F}$ of weighted cliques, we say G is $\mathcal{F}$-*free* if it contains no element of $\mathcal{F}$.

Definition 1.1. Given a set $\mathcal{F}$ of weighted cliques, we define for $n \in \mathbb{N}$

$$\beta_n(\mathcal{F}) := \max\{w(G) : \ |V(G)| = n, \ G \text{ is } \mathcal{F}\text{-free}\}.$$

Moreover, define

$$d(\mathcal{F}) := \lim_{n \to \infty} \beta_n(\mathcal{F}). \tag{1}$$

Problem 1.3. *Given* $\mathcal{F}$, *determine* $d(\mathcal{F})$.

Recall that $w(G) \cdot \frac{n}{n-1}$ is the average weight of edges in G. Using a standard averaging argument, it follows that for every $\mathcal{F}$, the sequence $\beta_n(\mathcal{F}) \cdot \frac{n}{n-1}$ is decreasing. Hence $\lim_{n \to \infty} \beta_n(\mathcal{F}) \cdot \frac{n}{n-1}$ exists and so does $d(\mathcal{F})$. As noted in the previous subsection, upper bounds on $d(\mathcal{F})$ for certain families $\mathcal{F}$ imply upper bounds on the Ramsey-Turán densities $\varrho_p(q)$.

A similar problem was investigated by Füredi and Kündgen [7]. Namely, for given integers k, r they studied the problem of determining the maximum weight of an integer-weighted graph where no k vertices induce a subgraph of total weight more than r.

A special case of Problem 1.3 can be stated as follows and it is a generalization of Turán's theorem. We call a function w from integers $2, 3, \ldots$ to $[0, 1]$ a *clique weighting*. Let G be a graph, e be an edge of G, and r be the order of a largest clique containing e. Then, we define

$$w(e) := w(r).$$

The general problem for a fixed w is to maximize $w(G)$ over all n-vertex graphs G, or over some family of graphs G.

Problem 1.4. *Let w be a clique weighting. Find $\alpha(n)$ for a fixed n, or at least determine its limit when n goes to infinity, where*

$$\alpha(n) := \max_{G:|V(G)|=n} w(G) \qquad and \qquad \alpha := \limsup_{n \to \infty} \alpha(n).$$

Since there are only finitely many n-vertex graphs, up to isomorphism, the maximum is well-defined. A particularly interesting clique weighting is

$$w_T(r) := \frac{r}{2(r-1)},$$

which we call *Turán weights*. Notice that Turán weights are defined such that for every $k \geq 2$, we get

$$\lim_{n \to \infty} w_T(T(n,k)) = \frac{1}{2},$$

and $w_T(T(n,k)) = \frac{1}{2}$, whenever n is divisible by k. We prove that Turán graphs are exactly the graphs maximizing $w_T(G)$ among all graphs G on n vertices as n goes to infinity.

Theorem 1.5. *For every graph G we have*

$$w_T(G) \leq \frac{1}{2}.$$

Theorem 1.5 was used by Balogh, Chen, McCourt and Murley [2] to obtain upper bounds for certain Ramsey-Turán problems. During the preparation of this work, Theorem 1.5 was independently proven by Malec and Tompkins [9].

We also prove a stability version of Theorem 1.5.

Theorem 1.6. *For every $\varepsilon > 0$, there exist n_0 and $\delta > 0$ such that if $w_T(G) \geq 1/2 - \delta$ and $n \geq n_0$, then G is in edit distance at most εn^2 from some Turán graph.*

The rest of the paper is structured as follows. In Section 2, we prove our results on Turán weights and discuss one application. In Section 3, we prove some results on the general weighted Turán problem which might be of independent interest. Finally, we prove Theorem 1.2 in Section 4.

2   Proofs of Theorems 1.5 and 1.6

Our proofs are based on the method of graph Lagrangians, first introduced by Motzkin and Straus [10]. Following [8], given an $m \times m$ matrix A, we define

$$g(A) := \max \left\{ \mathbf{u}^\mathsf{T} A \mathbf{u} \,\middle|\, \mathbf{u} = (u_1, \ldots, u_m)^\mathsf{T}, \ \sum_{i=1}^m u_i = 1, \ u_i \geq 0 \right\}.$$

The maximum is attained since it is taken over a compact set. A vector attaining the maximum is *optimal* for A. We say that A is *dense* if it has zero diagonal and for every $i \in [m]$, the matrix A' obtained by removing its ith row and column satisfies $g(A') < g(A)$. We use the following lemma from [8] which lists useful properties of optimal vectors of a dense matrix.

Lemma 2.1 (Liu, Reiher, Sharifzadeh, and Staden [8])**.** *Let $m \in \mathbb{N}$, $A = (a_{ij})$ be a dense symmetric $m \times m$ matrix with nonnegative entries, and $\mathbf{u}$ be optimal for A. Then,*

  a) A is positive, that is, $a_{ij} > 0$ for every $1 \leq i < j \leq m$,

  b) $u_i > 0$ for every $i \in [m]$,

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c) $\sum_{i\in[m]\setminus\{j\}} a_{ij}u_i = g(A)$, *for every* $j \in [m]$.

We note that in the original formulation of [8], the entries of A were restricted to the set $\{0, 1, \ldots, p\}$ for some positive integer p. However, the proof in [8] gives this stronger statement as well.

Proof of Theorem 1.5. Assume that G has at least one edge, otherwise the statement is trivial. Label the vertices of G by $\{v_1, \ldots, v_n\}$. Define a symmetric $n \times n$ matrix $A = (a_{ij})$ such that $a_{ij} = w_T(v_iv_j)$ if $v_iv_j \in E(G)$ and $a_{ij} = 0$ otherwise. Let A' be a minimal, by inclusion, principal submatrix of A maximizing $g(A')$. Since A' is minimal by inclusion, it is dense. It is also nonempty, since g is 0 on the empty matrix. Let $\mathbf{u}$ be optimal for A'. Let $K \subseteq [n]$ be the set of indices of vertices corresponding to the rows (and columns) of A' and write $k = |K|$. By Lemma 2.1, A' is positive, that is, the vertices indexed by K form a clique in G. Since $w_T(r)$ is decreasing in r, it follows that $a_{ij} \leq w_T(k)$ for all $i, j \in K$. Hence,

$$g(A) \leq g(A') \leq \sum_{i\in K} u_i \sum_{j\in K, j\neq i} u_j w_T(k) = w_T(k) \sum_{i\in K} u_i(1-u_i) = w_T(k)\left(1 - \sum_{i\in K} u_i^2\right) \leq w_T(k)\left(1 - \frac{1}{k}\right) = \frac{1}{2},$$

where we used that $\sum_{i\in K} u_i = 1$ and in the last inequality we used Jensen's inequality. On the other hand, by using $\mathbf{x} = (1/n, \ldots, 1/n)$, we obtain

$$w_T(G) = \frac{2}{n^2} \sum_{e\in E(G)} w_T(e) = \mathbf{x}^\mathsf{T} A \mathbf{x} \leq g(A) \leq \frac{1}{2},$$

completing the proof. □

Next, we prove the stability version as given by Theorem 1.6. Our proof is based on a careful analysis of the argument of [9].

Proof of Theorem 1.6. Fix $\varepsilon > 0$ and let $0 < \delta \ll \varepsilon$ be a small constant to be chosen implicitly later. Additionally, let $\delta' = 10\delta^{1/3}$. Let G be an n-vertex graph with sufficiently large n satisfying $w_T(G) \geq 1/2 - \delta$. We write $V = V(G)$. For $U \subseteq V$, define $W(U) := \sum_{e\in E(G[U])} w_T(e)$ and note that $W(V) = \frac{n^2}{2}w_T(G)$. For disjoint sets $A, B \subseteq V(G)$, define $W(A, B) := \sum_{e\in E(G[A,B])} w_T(e)$.

Define a sequence of graphs $G_1, G_2, \ldots$ as follows: set $G_1 = G$; for each $i \geq 1$, let C_i be a clique of maximum size in G_i and let $G_{i+1} := G_i \setminus C_i$. Let $t \geq 1$ be the minimum index such that $|V(G_{t+1})| \leq (1-\delta')n$. Let $A = \bigcup_{i=1}^t V(C_i)$ and $B = V(G) \setminus A$.

If $|B| \leq (\delta')^{1/3}n$, then $t = 1$. This implies that C_1 is a clique in G of size at least $(1 - (\delta')^{1/3})n$, and the statement trivially holds. Hence, from now on we may assume $|B| > (\delta')^{1/3}n$. We also have $|A| \geq \delta'n$ by the definition. Together they imply

$$|A||B| > \min\{(\delta')^{1/3}(1 - (\delta')^{1/3})n^2, \delta'(1 - \delta')n^2\} > 0.99\delta'n^2.$$

We use the following:

$$\frac{n^2}{2}w_T(G) = W(V) = W(A) + W(B) + W(A, B).$$

By Theorem 1.5, we have that $W(A) \leq |A|^2/4$ and $W(B) \leq |B|^2/4$. Using $|A| + |B| = n$ and our initial assumption that $w_T(G) \geq 1/2 - \delta$, we have

$$W(A, B) = W(V) - W(A) - W(B) \geq \frac{1}{4}\left((1 - 2\delta)n^2 - |A|^2 - |B|^2\right) = \frac{1}{4}\left(2(1 - 2\delta)|A||B| - 2\delta|A|^2 - 2\delta|B|^2\right)$$

$$\geq \left(\frac{1}{2} - \delta - \frac{\delta n^2}{|A||B|}\right)|A||B| \geq \left(\frac{1}{2} - \frac{1}{4}\delta'^2\right)|A||B|, \tag{2}$$

where in the last relation we used $\delta' = 10\delta^{1/3}$ and $|A||B| \geq 0.99\delta'n^2$.

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Fix $i \leq t, v \in B$ and let $N = N(v) \cap C_i$. Note that $N \cup \{v\}$ is a clique. Denote $|N|$ by $e(v, C_i)$. Hence, $N \neq C_i$ as otherwise $C_i \cup \{v\}$ is a clique larger than C_i in G_i. Each edge from v to C_i is in the clique $N \cup \{v\}$, so its weight is at most $w_T(|N| + 1)$. Hence,

$$W(\{v\}, C_i) \leq |N| w_T(|N| + 1) = \frac{|N| + 1}{2} = \frac{e(v, C_i) + 1}{2} \leq \frac{|C_i|}{2}.$$

Now, we can obtain an upper bound on $W(A, B)$:

$$W(A, B) = \sum_{i=1}^{t} \sum_{v \in B} W(C_i, \{v\}) \leq \sum_{i=1}^{t} \sum_{v \in B} \frac{1}{2}(e(v, C_i) + 1) \leq \sum_{i=1}^{t} \sum_{v \in B} \frac{1}{2}|C_i| = \frac{|A||B|}{2}. \tag{3}$$

Assume that for all $i \in [t]$, there are at least $\delta' n$ vertices $v \in B$ such that $e(v, C_i) + 1 \leq (1 - \delta')|C_i|$. Then, we can obtain a stronger bound on $W(A, B)$ as follows:

$$W(A, B) = \sum_{i=1}^{t} \sum_{v \in B} W(C_i, \{v\}) \leq \sum_{i=1}^{t} \sum_{v \in B} \frac{1}{2}(e(v, C_i) + 1) \leq \sum_{i=1}^{t} \frac{1}{2}\left((|B| - \delta' n)|C_i| + \delta' n (1 - \delta')|C_i|\right)$$

$$= \sum_{i=1}^{t} \frac{1}{2}|C_i|(|B| - \delta'^2 n) \leq \frac{1}{2}\left(1 - \delta'^2\right)|A||B|,$$

a contradiction with (2).

Hence, we can fix an $i \in [t]$ such that for all but at most $\delta' n$ vertices $v \in B$ it holds that $e(v, C_i) + 1 > (1 - \delta')|C_i|$. Let $B' \subseteq B$ denote the set of such vertices, so $|B'| \geq |B| - \delta' n \geq ((\delta')^{1/3} - \delta')n$.

Assume first that $|A| \geq 2\delta' n$. Then, $i = t = 1$ and $A = C_1$ is a clique. Let $uv \in G[B']$ be an arbitrary edge. Note that

$$|N(u) \cap N(v) \cap A| \geq (1 - 2\delta')|A| - 2,$$

hence

$$w_T(uv) \leq w_T((1 - 2\delta')|A|) < \frac{1}{2} + \frac{2}{|A|} < \frac{1}{2} + \frac{1}{\delta' n}.$$

Since there are at most $\delta' n^2$ edges touching $B \setminus B'$, using Theorem 1.5 and (3), we have

$$\frac{n^2}{2}\left(\frac{1}{2} - \delta\right) \leq W(V) \leq W(A) + W(A, B') + W(B') + \delta' n^2 \leq \frac{|A|^2}{4} + \frac{|A|(n - |A|)}{2} + W(B') + \delta' n^2$$

$$= W(B') + \frac{n|A|}{2} - \frac{|A|^2}{4} + \delta' n^2,$$

which implies

$$W(B') \geq \frac{n^2}{4} - \frac{n|A|}{2} + \frac{|A|^2}{4} - 2\delta' n^2 = \frac{(n - |A|)^2}{4} - 2\delta' n^2 \geq \left(\frac{1}{4} - 4(\delta')^{1/3}\right)|B'|^2,$$

where the last inequality holds because $|B'| \geq ((\delta')^{1/3} - \delta')n$. On the other hand, $W(B') \leq e(G[B']) \cdot (1/2 + 1/(\delta' n))$, so

$$e(G[B']) \geq \frac{(1/4 - 4(\delta')^{1/3})|B'|^2}{1/2 + 1/(\delta' n)} \geq \frac{1 - \varepsilon}{2} \cdot |B'|^2,$$

where we chose δ to be small enough with respect to ε. Hence,

$$e(G) \geq \binom{n}{2} - \frac{\varepsilon}{2}|B'|^2 - 2\delta' n^2 \geq (1 - \varepsilon)\binom{n}{2},$$

so G has edit distance at most ε to the complete graph and we are done.

Therefore, we may assume that $|A| < 2\delta'n$, so $|B| \geq (1 - 2\delta')n$ and $|B'| \geq (1 - 3\delta')n$. There are at most $3\delta'n^2$ edges touching $V \setminus B'$, so

$$W(B') \geq W(V) - 3\delta'n^2 > \frac{1}{4}(1 - 20\delta')n^2.$$

Define $k := |C_i|$. There are two cases:

Case 1: $k < 1/\delta'$. Then, for each $v \in B'$ we have $e(v, C_i) + 1 \geq (1 - \delta')k > k - 1$. That is, each $v \in B'$ is adjacent to all but one vertex in C_i. Hence, we can partition B' into sets $B_1', B_2', \ldots, B_k'$ according to which vertex of C_i is missing in the neighbourhood of $v \in B'$. Note that for every $j \in [k]$, B_j' is an independent set. Indeed, otherwise taking the two vertices which form an edge in B_j' and their $k - 1$ common neighbours in C_i, we obtain a clique of size $k + 1$ in G_i, a contradiction. Now, consider an edge $uv \in G[B']$. Note that $|N(u) \cap N(v) \cap C_i| = k - 2$, so uv lies in a clique of size k. Hence, $w_T(uv) \leq w_T(k)$. Therefore, $G[B']$ is a k-partite graph with

$$e(G[B']) \geq \frac{W(B')}{w_T(k)} \geq (1 - 20\delta')\frac{k-1}{k}\frac{n^2}{2}.$$

By the standard stability result for K_{k+1}-free graphs [5], it follows that $G[B']$ is in edit distance at most $\varepsilon/2n^2$ from $T(|B'|, k)$. Since $|B'| \geq (1 - 3\delta')n$, the result follows.

Case 2: $k \geq 1/\delta'$. Choose $u, v \in B'$ such that $uv \in E(G)$. Note that $|N(u) \cap N(v) \cap C_i| \geq (1 - 2\delta')k$, hence $w_T(uv) \leq w_T((1 - 2\delta')k + 2) < 1/2 + 1/k$. Hence,

$$e(G[B']) \geq \frac{W(B')}{1/2 + 1/k} \geq \frac{1 - 20\delta'}{1 + 2/k} \cdot \frac{n^2}{2} \geq (1 - \varepsilon/2)\frac{n^2}{2},$$

where we chose δ' to be sufficiently small compared to ε. Again, since $|B'| \geq (1 - 3\delta')n$, G is ε-close to the complete graph and the statement follows. $\square$

To capture the extremal examples, we compare rescaled weights. Let w be a clique weighting. We define a rescaling t_w as

$$t_w(r) := \frac{2(r-1)}{r}w(r).$$

Note that $t_{w_T}(r) = 1$ for all $r \in \mathbb{N}$. Observe that

$$t_w(r) = 2 \cdot \lim_{n \to \infty} w(T(n, r)),$$

i.e. it is the same as twice the weights of Turán graphs.

Corollary 2.2. *If w is a clique weighting, where $1 = \max_r t_w(r)$, then for every n-vertex graph G,*

$$w(G) \leq \frac{1}{2}.$$

Moreover, for every $\varepsilon_1, \varepsilon_2 > 0$, there exist n_0 and $\delta > 0$ such that if for each r either $t_w(r) = 1$ or $t_w(r) < 1 - \varepsilon_2$, $w(G) \geq 1/2 - \delta$, and $n \geq n_0$, then G is in edit distance at most $\varepsilon_1 n^2$ from some Turán graph $T(n, r)$, where r satisfies $t_w(r) = 1$.

Proof. Let w be a clique weighting with $1 = \max_r t_w(r)$. Observe that $w(r) \leq w_T(r)$ for all r as $t_w(r) \leq 1$. Hence, $w(G) \leq w_T(G) \leq 1/2$ by Theorem 1.5. If $w(G) \geq 1/2 - \delta$ then G is in edit distance $o(n^2)$ from a Turán graph $T(n, r)$ for some r, by Theorem 1.6. So, we can write $E(G) = (E(T(n, r)) \setminus F) \cup F'$, where $|F|, |F'| = o(n^2)$. Let $G' = T(n, r) \setminus F$, then $w(G') \geq 1/2 - \delta - o(1)$. Note that each edge in $T(n, r)$ is in $\Omega(n^{r-2})$ r-cliques and removing an edge from $T(n, r)$ removes $O(n^{r-2})$ r-cliques, so G' has $o(n^r)$ fewer r-cliques than $T(n, r)$. It follows that all but $o(n^2)$ edges of G' are contained in an r-clique, implying $w(r) = w_T(r)$, as claimed. $\square$
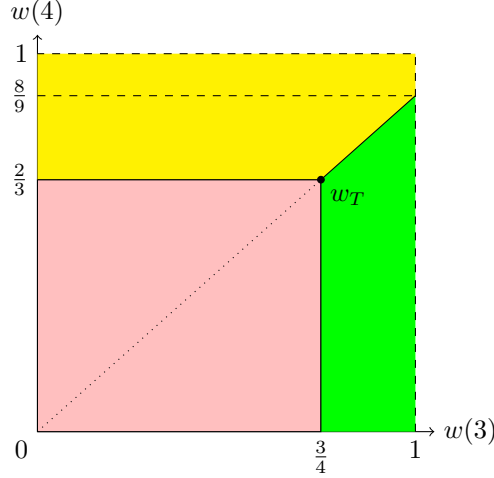
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Figure 2: Case with $w(2) = 1 = w_T(2)$ and point w_T depicted.

Now we demonstrate Corollary 2.2 on a small example of K_5-free graphs. Let w be a clique-weighting, where $w(2) = 1 = w_T(2)$, see Figure 2 for guidance. Corollary 2.2 implies the following (where we chose these particular numbers as they were used in [2]):

- If $w(3) \le 3/4$ and $w(4) \le 2/3$, then $T(n, 2)$ is extremal.

- If $w(3) \ge 3/4$ and $w(3) \ge \frac{9}{8}w(4)$, then $T(n, 3)$ is extremal.

- If $w(4) \ge 2/3$ and $w(3) \le \frac{9}{8}w(4)$, then $T(n, 4)$ is extremal.

3 Other weights

Recall the definitions of a weighted clique and of $d(\mathcal{F})$ from the beginning of Section 1.2. For $a \in [0, 1]$ we use K_r^a to denote the weighted clique (r, f) where $f(ij) = a$ for every $ij \in \binom{[r]}{2}$. Approaching Problem 1.3 of determining $d(\mathcal{F})$ for general $\mathcal{F}$ seems to be hard. However, the following lemma proves that if $\mathcal{F}$ contains K_{r+1}^0 (i.e. $\mathcal{F}$-free graphs are required to be K_{r+1}-free), then the asymptotically extremal graphs for β_n are blow-ups of a t-clique for some $t \le r$. We characterize these blow-ups as follows.

Definition 3.1. Given a positive integer t, a weighted clique (t, f) and a t-tuple $\mathbf{x} = (x_1, \ldots, x_t)$ of positive real numbers satisfying $\sum_{i=1}^{t} x_i = 1$, we define the $(n, t, f, \mathbf{x})$-*blow-up* as the following weighted graph G. Let $V(G) = V_1 \dot\cup V_2 \cup \ldots \dot\cup V_t$, where $|V_i| = \lfloor x_i n \rfloor$ for $1 \le i \le t - 1$ and $|V_t| = n - \sum_{i=1}^{t-1} |V_i|$. Let G be the complete t-partite graph with parts $V_1, \ldots, V_t$, where the weights of the edges between V_i and V_j are equal to $f(ij)$.

Lemma 3.1. *Let $\mathcal{F}$ be a finite set of weighted cliques containing K_{r+1}^0. Then, there exists a weighted clique (t, f) with $t \le r$ and a t-tuple $\mathbf{x} = (x_1, \ldots, x_t)$ of positive real numbers satisfying*

a) $\sum_{i=1}^{t} x_i = 1$,

b) $\sum_{j \ne i} x_j f(ij) = d(\mathcal{F})$ *for every $i \in [t]$,*

such that the $(n, t, f, \mathbf{x})$-blow-up G_n is $\mathcal{F}$-free and satisfies $w(G_n) = (1 - o(1))d(\mathcal{F})$.
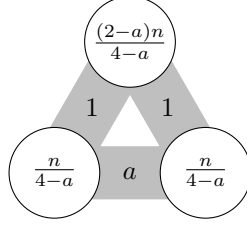
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Figure 3: The extremal weighted triangle from Theorem 3.2.

Proof. Let (G, w) be a weighted $\mathcal{F}$-free graph on vertex set $[n]$ with $w(G) = \beta_n(\mathcal{F})$. Let $A = (a_{ij})$ be the $n \times n$ symmetric matrix such that $a_{ij} = w(ij)$ and $a_{ii} = 0$. As in the proof of Theorem 1.5, we let A' be a minimal, by inclusion, principal submatrix of A maximizing $g(A')$ and apply Lemma 2.1 to obtain a clique K on $\{v_1, \ldots, v_t\}$ with $t \le r$ and positive real numbers $x_1, \ldots, x_t$ with $\sum_{i=1}^t x_i = 1$ and $\sum_{j \in [t], j \neq i} x_j w(v_i v_j) = g(A')$ for all $i \in [t]$. Then,

$$\beta_n(\mathcal{F}) = w(G) = \frac{2}{n^2} \sum_{e \in E(G)} w(e) \le g(A) \le g(A').$$

On the other hand, for $\mathbf{x} = (x_1, \ldots, x_t)$ and $f \colon \binom{[t]}{2} \to [0, 1]$ defined by $f(ij) = w(v_i v_j)$, the $(n, t, f, \mathbf{x})$-blow-up G_n is $\mathcal{F}$-free because G is $\mathcal{F}$-free and satisfies

$$w(G_n) \sim \sum_{1 \le i < j \le t} w(v_i v_j)(x_i n)(x_j n) \cdot \frac{2}{n^2} = \sum_{1 \le i \neq j \le t} w(v_i v_j) x_i x_j = g(A').$$

Therefore, for each $n \in \mathbb{N}$, we obtain parameters $t(n) \le r$, and $\mathbf{x}(n) \in (0, 1)^{t(n)}$ as above. Hence, there exist $t \le r$ and an infinite sequence a_n of positive integers such that $t(a_n) = t$ and $\lim_{n \to \infty} \mathbf{x}(a_n) = \mathbf{x}$, which implies the statement. $\qquad \square$

A triangle is (a, i)-*heavy* if at least i of its edges have weight strictly more than a. The following theorem determines the extremal K_4-free graphs with no $(a, 3)$-heavy triangles.

Theorem 3.2. *Let* $0 < a \le 1$ *and* $\mathcal{F} = \{K_3^a, K_4^0\}$, *then* $d(\mathcal{F}) = \frac{2}{4-a}$.

Proof. Applying Lemma 3.1 with $r = 3$ gives us a weighted clique (t, f) and a t-tuple $\mathbf{x}$ such that the $(n, t, f, \mathbf{x})$-blowup G_n is asymptotically optimal. If $t = 2$, then $w(G_n) = 1/2$. If $t = 3$, then one of the weights is at most a. Without loss of generality, assume $f(23) \le a$. Clearly, $w(G_n)$ is maximized when $f(12) = f(13) = 1$ and $f(23) = a$. Then we have $d(\mathcal{F}) = x_2 + x_3 = x_1 + a x_3 = x_1 + a x_2$ and $x_1 + x_2 + x_3 = 1$. Solving this system of equations, we obtain $x_1 = \frac{2-a}{4-a}, x_2 = x_3 = \frac{1}{4-a}$ and $d(\mathcal{F}) = \frac{2}{4-a} > 1/2$, as needed. See Figure 3 for a sketch of the construction. $\qquad \square$

Taking $a = 3/4$ we obtain the following corollary which was used in [2] to obtain several Ramsey-Turán type of results.

Corollary 3.3. *Let* G *be an* n-vertex K_4-free graph with a weight function $w : E(G) \to (0, 1]$. *If* G *contains no* $(3/4, 3)$-heavy triangles, then $w(G) \le \frac{8}{13} + o(1)$.

A K_q is a-*chubby*, if it has an edge of weight larger than a. For $r \ge q$, we determine the maximum weight of a K_{r+1}-free graph with no chubby K_q. In particular, this determines the extremal K_{r+1}-free graphs with no $(a, 1)$-heavy triangles. The possible extremal graphs are the r-clique with all edges of weight a and the $(q-1)$-clique with all edges of weight 1.

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Theorem 3.4. *Given integers $2 \le q \le r$ and $0 \le a \le 1$, let $f\colon \binom{[q]}{2} \to [0,1]$ be defined as $f(12) = a$ and $f(ij) = 0$ for all $\{i,j\} \ne \{1,2\}$. Denoting $\mathcal{F} = \{(q,f), K_{r+1}^0\}$, we have*

$$d(\mathcal{F}) = \max\{1 - 1/(q-1), a(1 - 1/r)\}.$$

Proof. Let $t, f, \mathbf{x}$ be the values obtained by applying Lemma 3.1 to the family $\mathcal{F}$ such that the $(n, t, f, \mathbf{x})$-blowup G_n satisfies $w(G_n) = (1 - o(1))d(\mathcal{F})$.

If $t \ge q$, then, since every edge of G_n is in a q-clique, we have that $f(ij) \le a$ for all $ij \in \binom{[t]}{2}$ and $w(G_n)$ is maximized if equality holds for all i, j and $t = r$. In that case, $w(G_n) \sim a(1 - 1/r)$.

On the other hand, if $t < q$, then $w(G_n)$ is maximized by taking $t = q - 1$ and $f(ij) = 1$ for all $ij \in \binom{[t]}{2}$, so in this case $w(G_n) \sim 1 - 1/(q - 1)$. Taking the maximum of the above two cases finishes the proof. $\square$

Finally, we consider the problem of forbidding an $(r + 1)$-clique and an $(a, 2)$-heavy triangle. Here the extremal examples are the complete bipartite graph with all edges of weight 1 and a blowup of an r-clique in which the edges of weight 1 form a matching of size $\lfloor r/2 \rfloor$ and the other edges have weight a.

Theorem 3.5. *Fix an integer r and a real $0 \le a \le 1$. Define $h\colon \binom{[3]}{2} \to [0,1]$ as $h(12) = h(13) = a, h(23) = 0$. Denoting $\mathcal{F} = \{(3, h), K_{r+1}^0\}$, we have*

$$d(\mathcal{F}) = \begin{cases} \frac{1}{2}, & \text{if } a \le \frac{1}{2}, \\ a + \frac{1 - 2a}{r}, & \text{if } a > \frac{1}{2} \text{ and } r \text{ is even}, \\ \frac{a^2(r-1)}{a(r+1)-1}, & \text{if } a > \frac{1}{2} \text{ and } r \text{ is odd}. \end{cases}$$

Proof. Let $(n, t, f, \mathbf{x})$ be the values obtained by applying Lemma 3.1 to $\mathcal{F}$ and let G_n denote the $(n, t, f, \mathbf{x})$-blow-up. Note that we may assume that all edge weights $f(ij)$ are either a or 1, as otherwise we could increase the weight while preserving the property that the $(n, t, f, \mathbf{x})$-blow-up is $\mathcal{F}$-free.

Since G_n is $\mathcal{F}$-free, the edges ij with $f(ij) = 1$ form a matching. The weight of G_n is clearly maximized if this matching is of largest possible size. Without loss of generality, assume that $f(2\ell - 1, 2\ell) = 1$ for all $1 \le \ell \le \lfloor t/2 \rfloor$, and $f(ij) = a$ for all other pairs. By part b) in Lemma 3.1, it follows that $x_{2\ell-1} = x_{2\ell}$ for all $\ell \in [\lfloor t/2 \rfloor]$.

First, assume that t is even. Then, for all $i \in [t]$,

$$d(\mathcal{F}) = \sum_{j \ne i} x_j f(ij) = x_i + a \cdot (1 - 2x_i) = a + (1 - 2a)x_i. \tag{4}$$

If $a = 1/2$, then $d(\mathcal{F}) = 1/2$. Otherwise, (4) implies that $x_1 = x_2 = \ldots = x_t = 1/t$, so $d(\mathcal{F}) = a + \frac{1-2a}{t}$.

Now, assume that t is odd. From part b) of Lemma 3.1, we obtain for all $i \in [t - 1]$

$$d(\mathcal{F}) = x_i + a(1 - 2x_i) = a + (1 - 2a)x_i,$$

If $a = 1/2$, it follows that $d(\mathcal{F}) = 1/2$. Otherwise, we obtain $x_1 = x_2 = \ldots = x_{t-1}$ and so $x_t = 1 - (t - 1)x_1$, which implies

$$d(\mathcal{F}) = a(1 - x_t) = a(t - 1)x_1 \le a.$$

If $a = 1/2$, then we have seen that $d(\mathcal{F}) = 1/2$. If $a < 1/2$, we obtain $d(\mathcal{F}) \ge 1/2$ by taking $t = 2$ and $d(\mathcal{F}) \le 1/2$ since $d(\mathcal{F})$ is clearly nondecreasing in a and we have $d(\mathcal{F}) = 1/2$ if $a = 1/2$. We conclude that $d(\mathcal{F}) = 1/2$ if $a \le 1/2$ and so from now on we assume that $a > 1/2$. Combining the two equations, we obtain $x_1 = \frac{a}{a(t+1)-1}$, which is well-defined since $a(t + 1) - 1 \ge 1/2 \cdot 3 - 1 > 0$. We conclude that in this case

$$d(\mathcal{F}) = a(t - 1)x_1 = \frac{a^2(t - 1)}{a(t + 1) - 1}.$$

We have shown that $d(\mathcal{F}) = 1/2$ if $a \le 1/2$. What is left to verify is that if $a > 1/2$, then $d(\mathcal{F})$ is maximized by taking $t = r$. It is enough to show that for any $a > 1/2$ and any even $t \ge 2$, we have

$$a + \frac{1 - 2a}{t} \le \frac{a^2 t}{a(t + 2) - 1} \le a + \frac{1 - 2a}{t + 2}.$$

This is a straightforward calculation so we omit the details. $\square$

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4   Ramsey-Turán density

Recall the definition of Ramsey-Turán density:

$$\varrho_p(q) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{RT_p(n, K_q, \varepsilon n)}{\binom{n}{2}},$$

where $RT_p(n, K_q, m)$ is the maximum number of edges in an n-vertex K_q-free graph in which every $m + 1$ vertices contain a clique of size p.

Given that the case $p = 2$ is fully resolved, we will focus on the case when $p \geq 3$. It is not hard to see that $\varrho_p(q) = 0$ when $q \leq p + 1$. We present a table of what is known about $\varrho_p(q)$, when both p and q are small. Note that if the lower and upper bounds are not matching, it is not clear which one is closer to the truth.

p \ q	5	6	7	8	9	10	11	12	13	14
3	H	S	E	H		E	H		E	H
4	0	H	S	S	E	H	★		E	H
5	0	0	S	S	S	S	E	★		
6	0	0	0	S	S	S	S	★	E	★

E means $q \equiv 1 \pmod p$. It was proved in [4] that the optimal construction is a complete $(q-1)/p$-partite graph, with a graph with sublinear α_p-independence number placed into each of the classes, i.e., $\varrho_p(q) = \frac{q-1-p}{q-1}$.

H means Theorem 1.4 in [8]:

$$\varrho_3(3t + 2) = \frac{5t - 4}{5t + 1} \qquad \text{and} \qquad \varrho_4(4t + 2) = \frac{7t - 6}{7t + 1}.$$

S means $q = p + \ell$, where $\ell \leq \min\{p, 5\}$. In [4], the upper bound $\varrho_p(q) \leq \frac{\ell - 1}{2p}$ was proved. For lower bounds see below.

★ We improve the upper bound.

Case $p = 3$:
For $q = 6$, a construction from [1] shows that $\varrho_3(6) \geq 1/4$. Joining this construction with independent sets (here joining means adding an independent set of vertices and all the cross-edges) iteratively (see also Figure 1) yields that $\varrho_3(9) \geq 4/7$ and $\varrho_3(12) \geq 7/10$. The upper bound $\varrho_3(9) \leq 3/5$ is from [4].

Case $p = 4$:
The lower bounds were proved both in [1] and [8]. The upper bounds for $q \leq 9$ are from [4], and we prove here $\varrho_4(11) \leq 4/7$:

$$\varrho_4(6) = 1/8, \quad \varrho_4(7) = 1/4, \quad 1/4 \leq \varrho_4(8) = 3/8, \quad \varrho_4(11) = 4/7.$$

Case $p = 5$:
The lower bounds were proved both in [8] for $q \leq 9$ and $q = 12, 13$, and in [1] for the others. The upper bounds for $q \leq 10$ are from [4], and we settle the case $q = 12$:

$$\varrho_5(7) = \frac{1}{10}, \quad \varrho_5(8) = \frac{1}{5}, \quad \frac{1}{4} \leq \varrho_5(9) \leq \frac{3}{10}, \quad \frac{1}{4} \leq \varrho_5(10) \leq \frac{2}{5}, \quad \varrho_5(12) = \frac{10}{19}, \quad \frac{5}{9} \leq \varrho_5(13), \quad \frac{4}{7} \leq \varrho_5(14).$$

Case $p = 6$:
The upper bounds are from [4] and the constructions are from [8]. The cases $q = 10, 11, 12$ are from [1]. In
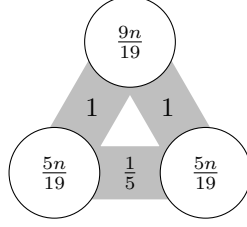
Figure 4: Sketch of a construction for $\varrho_5(12) \geq \frac{10}{19}$.

this paper, we prove the upper bounds for $q = 12, 14$. The construction for $q = 12$ is from [1], and for $q = 14$ it is from [8]:

$$\varrho_6(8) = \frac{1}{12}, \quad \varrho_6(9) = \frac{1}{6}, \quad \varrho_6(10) = \frac{1}{4}, \quad \frac{1}{4} \leq \varrho_6(11) \leq \frac{1}{3}, \quad \frac{1}{4} \leq \varrho_6(12) \leq \frac{5}{12}, \quad \varrho_6(14) = \frac{12}{23}.$$

Recall that Conjecture 1.1 was disproved by Liu, Reiher, Sharifzadeh, and Staden [8] by showing $\varrho_{16}(22) = 1/6 > 5/32 = \varrho_{16}^\star(22)$. While they resolved many cases when $p = 3$ and $p = 4$, they also asked what happens for larger p. While we solve only some sporadic cases, we think that it is interesting to determine, or at least upper bound some values, to see what constructions are or could be sharp.

Below, we recollect all the new results we prove in this paper.

Theorem 1.2. *The following bounds hold:*

$$\varrho_4(11) \leq \frac{4}{7}, \qquad \varrho_5(12) \leq \frac{10}{19}, \qquad \varrho_6(12) \leq \frac{5}{12}, \qquad \varrho_6(14) \leq \frac{12}{23}.$$

This implies $\varrho_5(12) = \frac{10}{19}$.

We tried to apply the method from the proof of Theorem 1.2 on $\varrho_3(12)$, $\varrho_5(13)$, $\varrho_5(14)$, $\varrho_4(12)$, and $\varrho_7(16)$. However, we would need more computational power to obtain reasonable bounds. The rest of the paper is dedicated to proving Theorem 1.2.

4.1 $\quad \varrho_5(12) = \frac{10}{19}$

Corollary 1.2 in [8] gives a construction showing $\varrho_5(12) \geq \frac{10}{19}$, see Figure 4. The existence of such a construction was suggested by Conjecture 1.1. In this section we show the construction is asymptotically best possible. The results in the rest of the section are proved in a similar way and only the differences from this section will be mentioned.

Proof outline. We only need to prove $\varrho_5(12) \leq \frac{10}{19}$. As mentioned in Section 1.1, we use the method developed by Liu, Reiher, Sharifzadeh, and Staden, see Chapter 5 in [8], that translates finding the upper bound of $\varrho_5(12)$ into a weighted Turán-type problem using Szemerédi's regularity lemma [4]. Then we use flag algebras to solve the weighted problem. We give an outline of the method, the interested reader may read Chapter 5 in [8] for a precise explanation.

First, fix a small $\varepsilon > 0$ and let G be a (large) graph with $\alpha_5(G) < \varepsilon n$, i.e., every set of at least εn vertices in G spans a copy of K_5. We want to show G has edge density at most $\frac{10}{19} + o(1)$.

Next, we apply the regularity lemma to G and obtain a cluster graph R. Recall that each vertex u in R corresponds to a cluster of vertices V_u in G. To simplify the notation, we let V_i to denote V_{v_i}. For vertices u and v in R, the *density* of uv, denoted by $w(u,v)$, is the edge density between V_u and V_v in G. Hence, we can view R as an edge-weighted graph with w being the weighting.

To turn the original problem into a weighted Turán problem, we first show there are some forbidden weighted cliques (see Section 3). To find them, we use the following embedding lemma:

Lemma 4.1. *Let $\varepsilon > 0$ and $t \in \mathbb{N}$. Then, there exists an $\varepsilon' > 0$ such that the following holds. Let $V_1, \ldots, V_t$ be clusters in an ε'-regular partition of a graph G with $\alpha_p(G) = o(n)$. Suppose that, for each $i < t$, we are given a permutation π_i of $\{i+1, \ldots, t\}$ and positive integers $s_{i,j}$, for $1 \leq i \leq j \leq t$ satisfying $s_{i,i} = p$, for every $i \in [t]$ and $s_{i,j} = \lceil s_{i,j-1}(d(V_i, V_{\pi_i(j)}) - \varepsilon/2) - \varepsilon/2 \rceil$. Then, G contains a clique of size $\sum_{i=1}^{t} s_{i,t}$.*

Proof sketch. Let $\varepsilon' \ll \varepsilon$ be chosen implicitly. Using standard cleaning arguments, we may assume that the pairs (V_i, V_j) are $(\varepsilon'', d(V_i, V_j) - \varepsilon'')$-superregular for some $\varepsilon'' \ll \varepsilon$. Recall that a pair (A, B) is (ϵ, δ)-*superregular* if it is ε-regular, $\deg(a) \geq \delta|B|$ for all $a \in A$ and $\deg(b) \geq \delta|A|$ for all $b \in B$. We will sequentially embed $s_{i,t}$ vertices into cluster V_i such that the embedded vertices form a clique and their common neighbourhood has linear size in each of the subsequent clusters. Fix some $i \in [t]$ and suppose we have already embedded vertices into clusters $V_1, \ldots, V_{i-1}$. By assumption, the embedded vertices have a linear sized common neighbourhood in V_i and since $\alpha_p(G) = o(n)$, there is a p-clique K in this set. Now, we find a subset of K which has a linear sized common neighbourhood in the clusters V_j for every $j > i$. To do this, we follow the "recipe" given to us by the permutation π_i and sizes $s_{i,j}$. Set $K_i = K$. Suppose $i < j \leq t$ and we have found $K_{j-1} \subseteq K$ such that the common neighbourhood of all vertices embedded into clusters $V_1, \ldots, V_{i-1}$ and K_{j-1} has linear size in each of the clusters $V_{\pi_i(i+1)}, \ldots, V_{\pi_i(j-1)}$. By superregularity, each vertex $v \in K_{j-1}$ has at least $d(V_i, V_{\pi_i(j)}) - \varepsilon''$ neighbours in $V_{\pi_i(j)}$. Recalling the definition of $s_{i,j}$ it follows that there is a set $K_j \subseteq K_{j-1}$ whose common neighbourhood in $V_{\pi_i(j)}$ is of linear size. In the end, we have embedded $s_{i,t}$ vertices into cluster V_i, yielding the desired clique. $\square$

Plugging in specific values into Lemma 4.1 can be somewhat opaque, so to make our proofs easier to follow, we write certain embedding rules. However, they are easily seen to be consequences of Lemma 4.1.

name/color	density interval	rule
1	$[0, \varepsilon)$	no embedding
2	$[\varepsilon, 1/5 + \varepsilon)$	any 1 vertex
3	$[1/5 + \varepsilon, 1/2 + \varepsilon)$	some 2 vertices
4	$[1/2 + \varepsilon, 3/5 + \varepsilon)$	any 2 vertices or some 3 vertices
5	$[3/5 + \varepsilon, 4/5 + \varepsilon)$	some 4 vertices
6	$[4/5 + \varepsilon, 1]$	any 5 vertices

Table 1: Names, density intervals, and brief rules.

In what follows, v_i, v_j denote two vertices in the cluster graph and $w(v_i, v_j)$ denotes the density $d(V_i, V_j)$ in G.

(R0) By assumptions on $\alpha_p(G)$, V_i contains K_5.

(R1) If $w(v_i, v_j) < \varepsilon$, then we cannot guarantee that vertices from both V_i and V_j can be used in an embedding of K_{12}.

(R2) If $w(v_i, v_j) \geq \varepsilon$, then arbitrary one vertex of V_i can be used in an embedding of K_{12}.

(R3) If $w(v_i, v_j) \geq 1/5 + \varepsilon$, then some two vertices of V_i can be used in an embedding of K_{12}.

(R4) If $w(v_i, v_j) \geq 1/2 + \varepsilon$, then arbitrary two vertices or some three vertices out of any $K_4 \subset K_5$ of V_i can be used in an embedding of K_{12}.

(R5) If $w(v_i, v_j) \geq 3/5 + \varepsilon$, then some 4 vertices of the K_5 could be used in an embedding of K_{12}.

(R6) If $w(v_i, v_j) \geq 4/5 + \varepsilon$, then the 5 vertices of the K_5 in V_i can be used in an embedding of K_{12}.

A brief summary of the rules is in Table 1. In justification for the rules, we want to show that there is a common neighborhood of linear size in V_j. If $w(v_i, v_j) = 0$, then there may be no edges at all between V_i and V_j, so a clique on at least 2 vertices cannot intersect both V_i and V_j. This gives (R1). For (R2), if

$w(v_i, v_j) > \varepsilon$, then any typical vertex in V_i has linearly many neighbors in V_j. For (R3) and (R5), first pick a K_5 in V_i by (R0). By the pigeonhole principle, 2 and 4 vertices respectively have a common neighborhood of linear size in V_j. In (R4) and (R6), two and 5, respectively vertices must have a linear common neighborhood in V_j. For the second part of (R4), applying the pigeonhole principle gives that some three out of any 4 vertices have a common neighborhood of linear size in V_j.

Notice that when considering V_i, we need to apply the rules towards all the remaining clusters. In particular, we can use a rule guaranteeing *some* subset such as (R3) or (R5) only once and the remaining rules need to be valid for *any* subset such as (R2) or (R6). For example, if we wish to embed 2 vertices in V_j, we can use (R3) to find 2 vertices X that work for one of the remaining classes but for the rest, we need to use (R4) that guarantees X has linear common neighborhood in all the remaining clusters.
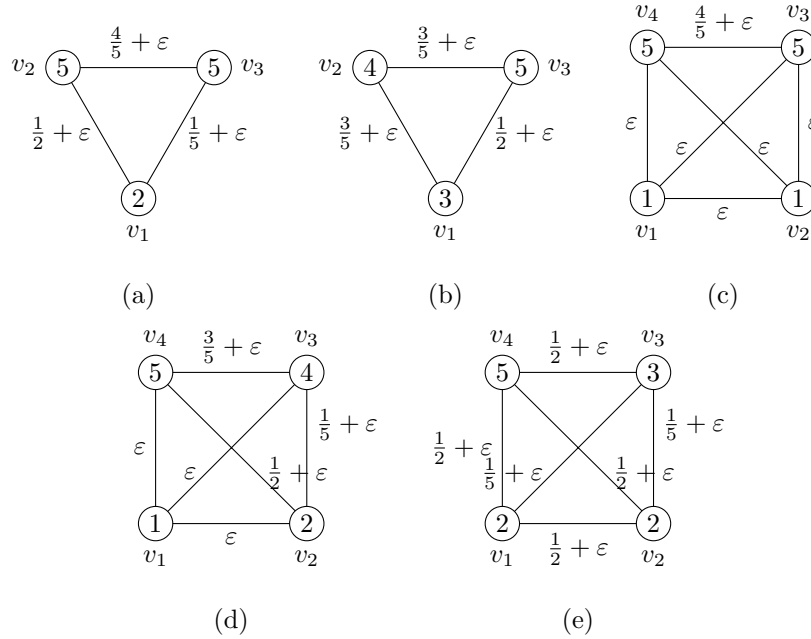

Figure 5: Forbidden weighted cliques $\mathcal{F}$ used in $\varrho_5(12) \leq \frac{10}{19}$. The label on edge denotes the edge weight. The number in vertex v_i indicates k_i, which is the number of vertices used from V_i to obtain K_{12}.

Let $\mathcal{F}$ be the weighted cliques depicted in Figure 5. Below we argue that if any of them appears in R, then we can find a K_{12} in G. To simplify the explanation, we skip explicitly mentioning applications of (R0):

(a) First we apply (R3) on $v_1 v_3$ to find 2 vertices in V_1. The same two vertices work also for $v_1 v_2$ by (R4). In V_2 we add a K_5 by (R6) on $v_2 v_3$. Finally, we find another K_5 in V_3.

(b) Using (R5) on $v_1 v_2$ we pick 4 vertices from V_1 that have a linear sized common neighborhood in V_2. Using (R4) on $v_1 v_3$, among these 4 vertices, we can choose 3 that have a linear sized common neighborhood in V_3. In V_2 we pick 4 vertices using (R5) on $v_2 v_3$ and V_3 contributes the remaining 5 vertices.

(c) Using (R2) around v_1 and v_2, we choose one vertex from each V_1 and V_2. From each V_3 and V_4 we choose five vertices, which is possible by (R6) on $v_3 v_4$.

(d) Using (R2) around v_1 we embed one vertex to V_1. Using (R3) on $v_2 v_3$, we choose some two vertices to V_2, and by (R4) the same two vertices work for $v_2 v_4$. In V_3 we add a K_4 by (R5) on $v_3 v_4$, and we find K_5 in V_4.
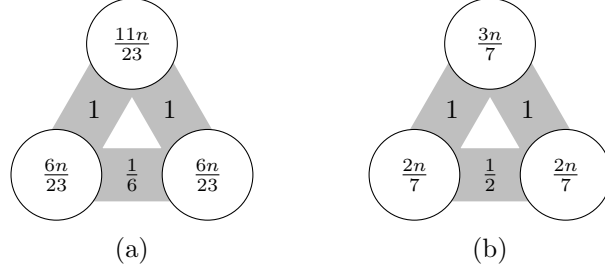
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Figure 6: Conjectured constructions from Conjecture 1.1 for (a) $\varrho_6(14) \geq \frac{12}{23}$ and (b) $\varrho_4(11) \geq \frac{4}{7}$.

(e) Using (R3) on $v_1 v_3$ we choose some two vertices from V_1. The same two vertices work for $v_1 v_2$ and $v_1 v_4$ by (R4). By (R3) on $v_2 v_3$, we embed some two adjacent vertices to V_2, and by (R4) the same two vertices work for $v_2 v_4$. By (R4) on $v_3 v_4$, we embed 3 vertices in V_3. Finally, we embed a K_5 in V_4.

Next we use flag algebras to find an upper bound of $d(\mathcal{F})$, see its definition in (1). We simplify the problem to an edge-colored extremal problem. Let H be a complete graph on the same vertex set as R. Let $c : E(H) \to [6]$ be a 6-edge-coloring defined according to Table 1. Notice that forbidding $\mathcal{F}$ in R gives a list of forbidden 6-edge-colored complete graphs $\mathcal{F}_c$. Each configuration in Figure 5 gives several forbidden configurations in $\mathcal{F}_c$.

Let c_i denote the density of edges of color i in H. Given the above 6-edge-colored problem, flag algebras can be applied to prove an upper bound

$$\frac{1}{5}c_2 + \frac{1}{2}c_3 + \frac{3}{5}c_4 + \frac{4}{5}c_5 + c_6 \leq \frac{10}{19} + o(1). \tag{5}$$

This implies $d(\mathcal{F}) \leq 10/19$. Notice that when splitting densities into intervals, we used the lower bounds of the intervals to obtain forbidden configurations but we used the upper bounds to calculate the weight. The calculation of (5) is computer assisted and it is too large to fit in the paper here. It can be downloaded at `http://lidicky.name/pub/rt/`. We use simple flag algebras approach, where we add a sum of squares to the left-hand side of (5) which yields the right-hand side. In addition to the bound 10/19, it would be possible to obtain a stability type result that the cluster graph is a blow-up of the graph depicted in Figure 4. □

4.2 $\varrho_6(14) \leq \frac{12}{23}$

Here $p = 6$ and $q = 14$, so in Conjecture 1.1 we get $t = 2, r = 0$. The three-partite construction depicted in Figure 6(a) gives $\frac{2p}{4p-1} = \frac{12}{23} \approx 0.5217391304$.

The proof of the upper bound in this section is analogous to the proof in Section 4.1. The main difference is that we use a different discretization of density intervals as in the table below, and different forbidden weighted cliques depicted in Figure 7.

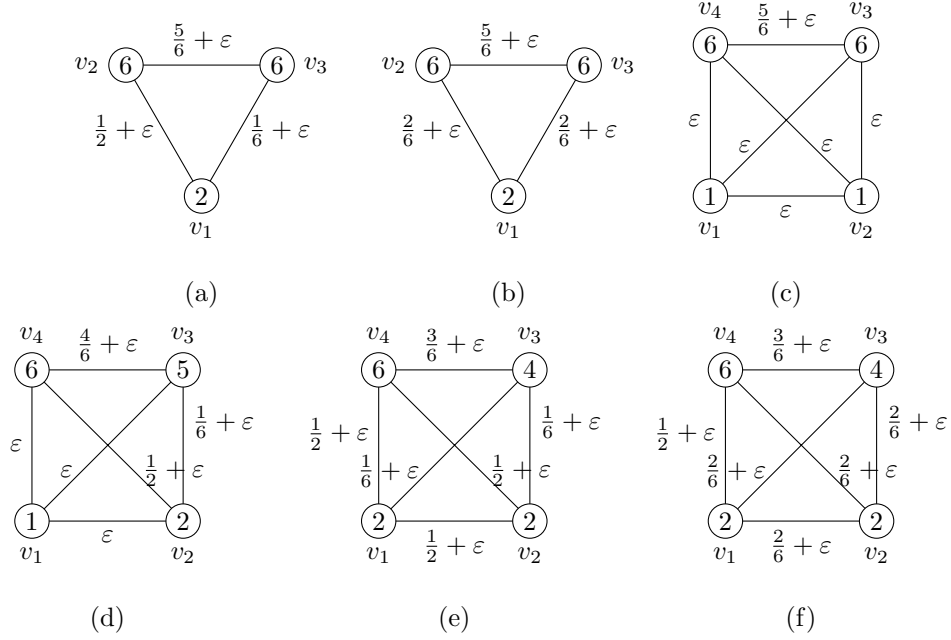| name/color	weight interval	rule
1	$[0, \varepsilon)$	no embedding
2	$[\varepsilon, 1/6 + \varepsilon)$	any 1 vertex
3	$[1/6 + \varepsilon, 2/6 + \varepsilon)$	some 2 vertices
4	$[2/6 + \varepsilon, 3/6 + \varepsilon)$	some 3 vertices
5	$[3/6 + \varepsilon, 4/6 + \varepsilon)$	any 2 vertices or some 4 vertices
6	$[4/6 + \varepsilon, 5/6 + \varepsilon)$	some 5 vertices
7	$[5/6 + \varepsilon, 1]$	any 6 vertices

Figure 7: Forbidden configurations used in $\varrho_6(14) \leq \frac{4}{7}$.

The proof is finished by using flag algebra calculation on 4 vertices. The computation is computer assisted and it can be downloaded at `http://lidicky.name/pub/rt/`.

4.3 $\varrho_4(11) \leq \frac{4}{7}$

Here $p = 4$ and $q = 11$, so in Conjecture 1.1 we get $t = 2, r = 1$. The three-partite construction depicted in Figure 6(b) gives $\frac{4}{7}$.

The proof of the upper bound in this section is analogous to the proof in Section 4.1. The main difference is that we use a different discretization of density intervals as in the table below, and different forbidden weighted cliques depicted in Figure 8.

name/color	density interval	rule
1	$[0, \varepsilon)$	no embedding
2	$[\varepsilon, 1/4 + \varepsilon)$	any 1 vertex
3	$[1/4 + \varepsilon, 1/2 + \varepsilon)$	some 2 vertices
4	$[1/2 + \varepsilon, 3/4 + \varepsilon)$	any 2 vertices or some 3 vertices
5	$[3/4 + \varepsilon, 1]$	any 4 vertices

The proof is finished by using flag algebra calculation on 5 vertices. The computation is computer assisted and it can be downloaded at `http://lidicky.name/pub/rt/`.

4.4 Upper bounds for $\varrho_p(p+6)$

In this section we show

$$\varrho_p(p+6) \leq \begin{cases} \frac{5}{2p} & \text{if } p \leq 12 \\ \frac{8}{3p} & \text{otherwise.} \end{cases}$$
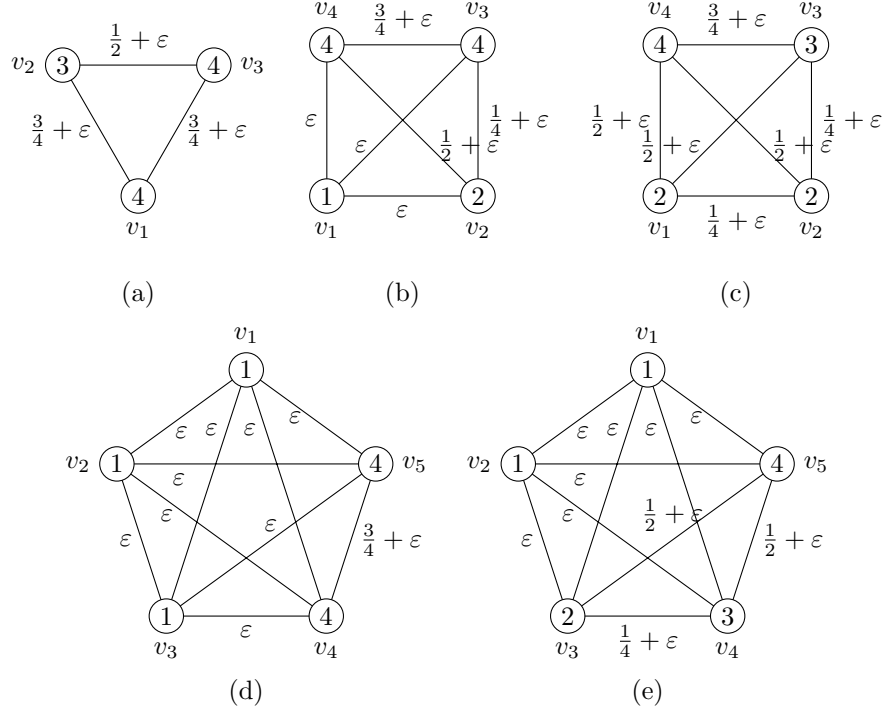
Figure 8: Forbidden configurations used in $\varrho_4(11) \leq \frac{4}{7}$.

We mentioned earlier that $\varrho_3(9) \leq 3/5$, which is better than our bound. Note that for $p = 4$, [8] proved $\varrho_4(10) = \frac{8}{15}$, which is also better than our bound. For $p = 5$, Erdős [4] has $\varrho_5(11) = \frac{5}{10}$, which is the same as our bound. The case $p \geq 13$ is implied by Theorem 1.5 from [8] by setting $s = 4$ and $t = 3$. We reprove it here using our method. In the following we assume $p \geq 6$.

The two conjectured extremal constructions are a balanced bipartite graph with edge density $\frac{5}{p}$ between the two parts giving edge density $\frac{5}{2p}$, and a balanced tripartite graph with edge density $\frac{4}{p}$ between every two parts, giving edge density $\frac{8}{3p}$. Notice that these are not necessarily realizable since one has to fill in the parts graphs, where every independent set of αn vertices contain K_p but there is no K_{p+6} in the resulting graph. It is not obvious how to make such a construction.
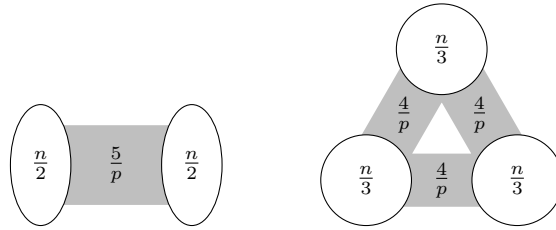
Notice that the tripartite construction has a higher edge density. Below, we rule out the tripartite construction for $p \leq 12$, while Liu, Reiher, Sharifzadeh, and Staden [8] actually constructed it for $p = 16$, and commented that their method should work for some other values of p.

Now we focus on the upper bound. First observe that there is no edge in the cluster graph with weight at least $5/p + \varepsilon$. If there was one, we could find $6 + p$ vertices forming a clique, which is a contradiction.

The rest of proof of the upper bound is again analogous to the proof in Section 4.1. We discretize the

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densities according to the following table. Notice that the table works for all p since we have an upper bound of $5/p + \varepsilon$ on the densities.

name/color	weight interval	rule
1	$[0, \varepsilon)$	no embedding
2	$[\varepsilon, 1/p + \varepsilon)$	any 1 vertex
3	$[1/p + \varepsilon, 2/p + \varepsilon)$	some 2 vertices
4	$[2/p + \varepsilon, 3/p + \varepsilon)$	some 3 vertices
5	$[3/p + \varepsilon, 4/p + \varepsilon)$	some 4 vertices
6	$[4/p + \varepsilon, 5/p + \varepsilon)$	some 5 vertices

The forbidden configurations used for showing $\varrho_p(p+6) \leq \frac{8}{3p}$ are depicted in Figure 9.
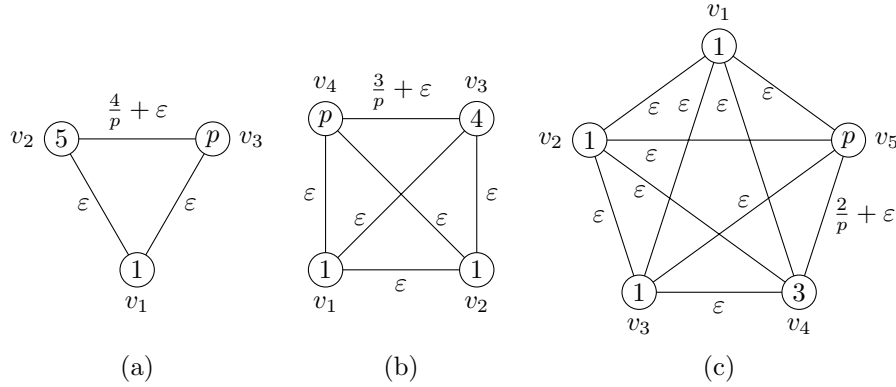
Figure 9: Forbidden configurations used in $\varrho_p(p+6) \leq \frac{8}{3p}$.

For $p \leq 12$, we can add one more forbidden configuration to the list that results in an improved upper bound, see Figure 10. The only trick is with v_1. First, we pick 4 vertices that have linear common neighborhood in V_3. From these 4 vertices, we can pick 2 by pigeonhole principle that have linear common neighborhood in V_2. These are the two vertices we keep. The second step works only if $4 \cdot \left(\frac{3}{p} + \varepsilon\right) > 1$, which gives the restriction $p \leq 12$. This provides bound $\varrho_p(p+6) \leq \frac{5}{2p}$. The computation is computer assisted and it can be downloaded at `http://lidicky.name/pub/rt/`.
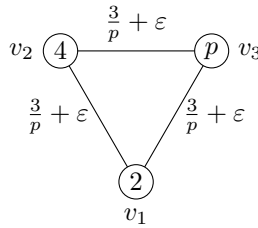

Figure 10: An extra forbidden configuration in addition to Figure 9 used to show $\varrho_p(p+6) \leq \frac{5}{2p}$.

Acknowledgment

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