On Choosability with Separation of Planar Graphs with Forbidden Cycles

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Abstract

We study choosability with separation which is a constrained version of list coloring of graphs. A (k, d)-*list assignment* L of a graph G is a function that assigns to each vertex v a list $L(v)$ of at least k colors and for any adjacent pair xy, the lists $L(x)$ and $L(y)$ share at most d colors. A graph G is (k, d)-choosable if there exists an L-coloring of G for every (k, d)-list assignment L. This concept is also known as choosability with separation. We prove that planar graphs without 4-cycles are $(3, 1)$-choosable and that planar graphs without 5-cycles and 6-cycles are $(3, 1)$-choosable. In addition, we give an alternative and slightly stronger proof that triangle-free planar graphs are $(3, 1)$-choosable.

1 Introduction

Given a graph G, a *list assignment* L is a function on $V(G)$ that assigns to each vertex v a list $L(v)$ of (*available*) colors. An *L-coloring* is a vertex coloring φ such that $\varphi(v) \in L(v)$ for each vertex v and $\varphi(x) \neq \varphi(y)$ for each edge xy. A graph G is said to be k-choosable if there is an L-coloring for each list assignment L where $|L(v)| \geq k$ for each vertex v. The minimum such k is known as the choosability of G, denoted $\chi_\ell(G)$. A graph G is said to be (k, d)-choosable if there is an L-coloring for each list assignment L where $|L(v)| \geq k$ for each vertex v and $|L(x) \cap L(y)| \leq d$ for each edge xy.

This concept is known as choosability with separation, since the second parameter may force the lists on adjacent vertices to be somewhat separated. If G is (k, d)-choosable, then G is also (k', d')-choosable for all $k' \geq k$ and $d' \leq d$. A graph is (k, k)-choosable if and only if it is k-choosable. Clearly, all graphs are $(k, 0)$-choosable for $k \geq 1$. Thus, for a graph G and each $1 \leq k < \chi_\ell(G)$, there is some threshold $d \in \{0, \ldots, k-1\}$ such that G is (k, d)-choosable but not $(k, d+1)$-choosable.

This concept of choosability with separation was introduced by Kratochvíl, Tuza, and Voigt [4]. They used the following, more general definition. A graph G is (p, q, r)-*choosable*, if for every list assignment L with $|L(v)| \geq p$ for each $v \in V(G)$ and $|L(u) \cap L(v)| \leq p - r$

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whenever u, v are adjacent vertices, G is q-tuple L-colorable. Since we consider only $q = 1$, we use a simpler notation. They investigate this concept for both complete graphs and sparse graphs. The study of dense graphs were extended to complete bipartite graphs and multipartite graphs by Füredi, Kostochka, and Kumbhat [2, 3].

Thomassen [6] proved that planar graphs are 5-choosable, and hence they are $(5, d)$-choosable for all d. Voigt [9] constructed a non-4-choosable planar graph, and there are also examples of non-$(4, 3)$-choosable planar graphs. Kratochvíl, Tuza, and Voigt [4] showed that all planar graphs are $(4, 1)$-choosable. The question of whether all planar graphs are $(4, 2)$-choosable or not was raised in the same paper and it still remains open.

Voigt [8] also constructed a non-3-choosable triangle-free planar graph. Škrekovski [10] observed that there are examples of triangle-free planar graphs that are not $(3, 2)$-choosable, and posed the question of whether or not every planar graph is $(3, 1)$-choosable; Kratochvíl, Tuza and Voigt [4] proved the following partial case of this question:

Theorem 1. *Every triangle-free planar graph is* $(3, 1)$*-choosable.*

We strengthen Theorem 1 by showing an alternative proof that uses a method developed by Thomassen; we also use this method to prove Theorem 2 below. Our inspiration was Thomassen's proof [7] that every planar graph of girth 5 is 3-choosable. We also prove the following two different partial cases:

Theorem 2. *Every planar graph without* 4*-cycles is* $(3, 1)$*-choosable.*

Theorem 3. *Every planar graph without* 5*-cycles and* 6*-cycles is* $(3, 1)$*-choosable.*

These results are similar in nature to other results on the choosability of planar graphs when certain cycles are forbidden (see a survey of Borodin [1]). One of the motivations is Steinberg's Conjecture that states that all planar graphs containing no 4- or 5-cycles are 3-colorable [5]. We construct a planar graph without cycles of length 4 and 5 that is not $(3, 2)$-choosable, to show that Steinberg's Conjecture cannot be extended to $(3, 2)$-choosability.

Theorems 1 and 2 are shown in Sections 2 and 3, respectively. Theorem 3 uses a discharging technique, and is showed in Section 4.

1.1 Preliminaries and Notation

Always L is a list assignment on the vertices of a graph G. In our proofs of Theorems 1 and 2, we use list assignments where vertices can have lists of different sizes. A $(*, 1)$-*list assignment* is a list assignment L where $|L(v)| \geq 1$ and $|L(u) \cap L(v)| \leq 1$ for every pair of adjacent vertices u, v. A vertex v is an Ld*-vertex* when $|L(v)| = d$.

Given a graph G and a cycle $K \subset G$, an edge uv of G is a *chord* of K if $u, v \in V(K)$, but uv is not an edge of K. For an integer $k \geq 2$, a path $v_0 v_1 \dots v_k$ is a k*-chord* if $v_0, v_k \in V(K)$ and $v_1, \dots, v_{k-1} \notin V(K)$. If G is a plane graph, then let $\mathrm{Int}_K(G)$ be the subgraph of G consisting of the vertices and edges drawn inside the closed disc bounded by K, and let $\mathrm{Ext}_K(G)$ be the subgraph of G obtained by removing all vertices and edges drawn inside the open disc bounded by K. In particular, $K = \mathrm{Int}_K(G) \cap \mathrm{Ext}_K(G)$.

Note that each k-chord of K belongs to exactly one of $\mathrm{Int}_K(G)$ or $\mathrm{Ext}_K(G)$. If the cycle K is the outer face of G and Q is a k-chord of K, then let C_1 and C_2 be the two cycles in $K \cup Q$ that contain Q. Then the subgraphs $G_1 = \mathrm{Int}_{C_1}(G)$ and $G_2 = \mathrm{Int}_{C_2}(G)$ are the *Q-components* of G.

A graph G is *H-free* if it does not contain a copy of H as a subgraph.

2 Forbidding 3-cycles

In this section, we prove Theorem 1 as a corollary of the following theorem. Observe that any $(3, 1)$-list assignment on a triangle-free plane graph satisfies the conditions of the following theorem.

Theorem 4. *Let G be a triangle-free plane graph with outer face F with a subpath $P \subset F$ containing at most two vertices, and let L be a $(*, 1)$-list assignment such that the following conditions are satisfied:*

(i) $|L(v)| \geq 3$ *for* $v \in V(G) \setminus V(F)$,

(ii) $|L(v)| \geq 2$ *for* $v \in V(F) \setminus V(P)$,

(iii) $|L(v)| = 1$ *for* $v \in V(P)$,

(iv) no two vertices with lists of size two are adjacent in G,

(v) the subgraph induced by $V(P)$ *is* L *colorable.*

Then G is L-colorable.

Proof. Let G be a counterexample where $|V(G)| + |E(G)|$ is as small as possible. By the minimality of G, we assume that $|L(u) \cap L(v)| = 1$ for every edge $uv \in E(G) \setminus E(P)$. If otherwise, then we can remove the edge uv to obtain an L-coloring of $G - uv$, which is also an L-coloring of G. It is also clear that G is connected.

We quickly prove that G is 2-connected. Suppose v is a cut-vertex of G. There exist nontrivial connected induced subgraphs G_1 and G_2 of G such that $G_1 \cup G_2 = G$ and $V(G_1) \cap V(G_2) = \{v\}$. Assume by symmetry that $P \subseteq G_1$. By the minimality of G, there exists an L-coloring φ of G_1. Let L' be the list assignment on $V(G_2)$ where $L'(u) = L(u)$ if $u \neq v$ and $L'(v) = \{\varphi(v)\}$; the lists L' satisfy the hypothesis on G_2. By the minimality of G, the graph G_2 has an L'-coloring ψ where $\psi(v) = \varphi(v)$, so φ and ψ form an L-coloring of G.

Since G is 2-connected, the outer face is bounded by a cycle. In the following claims, we prove that the cycle on F does not have chords or certain types of 2-chords.

Claim 4.1. *F does not contain any chords.*

Proof. Suppose for the sake of contradiction that $Q = uv$ is a chord of F. Let G_1 and G_2 be the two Q-components of G. Assume by symmetry that $P \subseteq G_1$. By the minimality of G, there exists an L-coloring φ of G_1. Let L' be the list assignment on $V(G_2)$ where for

$x \in V(G_2)$, $L'(x) = \varphi(x)$ if $x \in \{u, v\}$ and $L'(x) = L(x)$ otherwise. By the minimality of G, there exists an L'-coloring ψ of G_2 with $\psi(u) = \varphi(u)$ and $\psi(v) = \varphi(v)$; together ψ and φ form an L-coloring of G. □

A 2-chord $v_0v_1v_2$ of F is *bad* if v_0 or v_2 is an $L2$-vertex. An $L3$-vertex $x \in V(F)$ is *good* if there is no bad 2-chord of F containing x.

Claim 4.2. *G has a good vertex.*

Proof. Suppose that F has no good vertex, so all $L3$-vertices in F are contained in a bad chord. Since G is 2-connected and triangle-free, $|V(F)| \geq 4$. Hence F contains at least one $L3$-vertex. Among all $L3$-vertices in F, let v_0 be an $L3$-vertex with a bad 2-chord $Q = v_0v_1v_2$ such that the size of the Q-component G_2 not containing P is minimized.

Let u be the neighbor of v_2 on F that is in G_2. Since G is triangle-free, the vertices u and v_0 are distinct. Since Q is a bad 2-chord, v_2 is an $L2$-vertex and hence u is an $L3$-vertex. Since F has no good $L3$-vertices, there is a bad 2-chord $Q' = uu_1u_2$ of F where u_2 is an L2-vertex. Since G is triangle-free, $u_1 \neq v_1$. Therefore, Q' is contained in G_2 and the Q'-component not containing P is properly contained within G_2, contradicting our extremal choice. □

Let $v_0v_1v_2$ be a path in F where v_1 is a good vertex. There exists a color c in $L(v_1)$ that does not appear in $L(v_0) \cup L(v_2)$. We will color v_1 with c and extend that coloring to $G - v_1$. Let $G' = G - v_1$, and let L' be the list assignment on $V(G')$ where $L'(u) = L(u) \setminus \{c\}$ for vertices u adjacent to v_1 in G, and $L'(u) = L(u)$ otherwise.

The neighbors of v_1 are $L'2$-vertices in G', and we verify that G' satisfies our hypotheses. Since G is triangle-free, the neighbors of v_1 form an indepenent set. Since v_1 is a good vertex, the $L'2$-vertices in G' form an independent set. By minimality of G, the graph G' has an L'-coloring φ. This L'-coloring φ extends to an L-coloring of G by assigning $\varphi(v_1) = c$. □

3 Forbidding 4-cycles

In this section, we prove Theorem 2 using a strengthened hypothesis. Observe that any $(3, 1)$-list assignment on a C_4-free planar graph satisfies the conditions of the following theorem.

Theorem 5. *Let G be a C_4-free plane graph with outer face F with a subpath P of F containing at most three vertices, and let L be a $(*, 1)$-list assignment such that the following conditions are satisfied:*

(i) $|L(v)| \geq 3$ *for* $v \in V(G) \setminus V(F)$,

(ii) $|L(v)| \geq 2$ *for* $v \in V(F) \setminus V(P)$,

(iii) $|L(v)| = 1$ *for* $v \in V(P)$,

(iv) no two $L2$-vertices are adjacent in G,

(v) the subgraph induced by $V(P)$ is L colorable,

(vi) no vertex with list of size two is adjacent to two vertices of P.

Then G is L-colorable.

Proof. Let G be a counterexample where $|V(G)| + |E(G)|$ is as small as possible. Moreover, we assume that the sum of the sizes of the lists is also as small as possible subject to the previous condition. By the minimality of G, we assume that for every edge $uv \in E(G)\setminus E(P)$, $|L(u) \cap L(v)| = 1$. If otherwise, then we can remove the edge uv to obtain an L-coloring of $G - uv$, which is also an L-coloring of G. It is also clear that G is connected.

Moreover, we show G is 2-connected. Suppose v is a cut-vertex of G. There exist nontrivial connected induced subgraphs G_1 and G_2 such that $G_1 \cup G_2 = G$ and $V(G_1) \cap V(G_2) = \{v\}$. Suppose P is contained within exactly one of G_1 or G_2; by symmetry $P \subseteq G_1$. By the minimality of G, there exists an L-coloring φ of G_1. Let L' be the list assignment on $V(G_2)$ where $L'(u) = L(u)$ if $u \neq v$ and $L'(v) = \{\varphi(v)\}$. By the minimality of G, there exists an L'-coloring of G_2 and this coloring combined with φ gives an L-coloring of G. When P is not contained within only one of G_1 or G_2, we have $v \in V(P)$. By the minimality of G, both G_1 and G_2 are L-colorable and these colorings agree on v which gives an L-coloring of G.

In Claims 5.1 through 5.5, we determine certain structural properties of our counterexample G. A vertex v is a *middle vertex* if it has degree two in P. Observe that since G is C_4-free, any two vertices have at most one common neighbor.

Claim 5.1. *G does not contain a triangle with nonempty interior.*

Proof. Suppose not and assume $T = pqr$ is a triangle with nonempty interior in G. Let $G_1 = \mathrm{Ext}_T(G)$ and $G_2 = \mathrm{Int}_T(G)$. Since T has nonempty interior, $|V(G_1)| < |V(G)|$, and there exists an L-coloring φ of G_1. Let G' be obtained from G_2 by removing the edge rp and let L' be a list assignment on $V(G')$ where $L'(v) = \{\varphi(v)\}$ if $v \in \{p, q, r\}$ and $L'(v) = L(v)$ otherwise. The hypothesis applies to G' and L' with pqr as the path on three precolored vertices on the outer face of G'. Since $|E(G')| < |E(G)|$, there exists an L'-coloring ψ of G' which combined with φ forms an L-coloring of G. This contradicts G being a counterexample. □

If $|V(F)| \leq 4$, then $|V(F)| = 3$ since G contains no 4-cycles. By Claim 5.1, $G = F$ and it is easy to check Theorem 5 for graphs with at most three vertices. Thus, $|V(F)| \geq 5$.

A chord $Q = uv$ is *bad* if one of the Q-components is a triangle uvx where $|L(x)| = 2$. Otherwise, the chord Q is *good*.

Claim 5.2. *F contains only bad chords.*

Proof. For a good chord $Q = uv$, let G_1 and G_2 be the Q-components such that $|V(G_1) \cap V(P)| \geq |V(G_2) \cap V(P)|$. If F contains a good chord, select a good chord Q that minimizes $|V(G_2)|$. Since Q is good, the vertices u and v are at distance at least three apart in the path $F \cap G_2$. Assume $v \notin V(P)$.

By the minimality of G, there exists an L-coloring φ of G_1. Let L' be the list assignment on $V(G_2)$ where $L'(x) = \{\varphi(x)\}$ if $x \in \{u, v\}$ and $L'(x) = L(x)$ otherwise. Since uv is a chord, Since G_2 contains fewer vertices of P than G_1, the graph G_2 has at most three $L'1$-vertices, and they form a path of length at most two on the outer face of G_2.

Since we only changed the lists on u and v in G_2, the $L'2$-vertices remain an independent set. The only condition that remains to be verified is that every $L'2$-vertex in G_2 has at most one $L'1$-neighbor.

Suppose there exists an $L'2$-vertex $x \in V(G_2)$ adjacent to two $L'1$-vertices. Since x is not adjacent to two $L2$-vertices, one of these vertices must be v, which is not an $L1$-vertex. Since G has no 4-cycles, these two $L'1$-vertices must be adjacent, so x is adjacent to u and v. Since $|L(x)| = 2$, if either ux or vx is a chord, then it must be a good chord, so this contradicts the choice of Q. Hence both ux and vx are edges of F. Moreover, Claim 5.1 implies that G_2 is exactly the triangle uvx, which contradicts that Q is a good chord.

Hence there exists an L'-coloring ψ of G_2 that agrees with φ on Q, and these colorings together form an L-coloring of G. □

Claim 5.3. *F contains only bad chords uv where $u, v \notin V(P)$.*

Proof. Suppose for a contradiction that uv is a bad chord and $u \in V(P)$. Let $z \in V(F)$ be a common neighbor of u and v forming the bad chord. Since $|L(u)| = 1$, $L(u) \subset L(v)$ and $L(u) \subset L(z)$. Hence $L(v) \cap L(z) = L(u)$. By the minimality of G, there exists an L-coloring of $G - vz$. However, it is also an L-coloring of G. □

A 2-chord $Q = v_0v_1v_2$ of a cycle K is *separating* if $v_0v_2 \notin E(K)$. We now eliminate the possibility of F containing certain separating 2-chords.

Claim 5.4. *F does not contain a separating 2-chord $v_0v_1v_2$ where $|L(v_2)| = 2$ and v_0 is not a middle vertex.*

Proof. For a separating 2-chord $Q = v_0v_1v_2$ where v_2 is an $L2$-vertex and v_0 is not a middle vertex, let G_1 and G_2 be the Q-components of G where G_1 contains the vertices of P. If such a 2-chord exists, select Q to minimize $|V(G_2)|$.

By the minimality of G, there exists an L-coloring φ of G_1. Let L' be the list assignment on G_2 where $L'(v_i) = \{\varphi(v_i)\}$ for $i \in \{0, 1, 2\}$ and $L'(x) = L(x)$ for $x \in V(G_2) \setminus V(Q)$. The $L'1$-vertices of G_2 are exactly v_0, v_1, and v_2.

Since the $L'2$-vertices are also $L2$-vertices, the hypothesis holds for G_2 and L' as long as every $L'2$-vertex in G_2 has at most one neighbor in Q. Since v_2 in an $L2$-vertex it is not adjacent to any other $L2$-vertices. If some $L'2$-vertex x is adjacent to both v_1 and v_0, then the separating 2-chord v_2v_1x contradicts our extremal choice of Q.

Hence by the minimality of G there exists an L'-coloring ψ of G_2 which agrees with φ on Q and together these colorings form an L-coloring of G. □

Claim 5.5. *F does not contain a separating 2-chord $v_0v_1v_2$ where $|L(v_2)| = 3$, $v_0 \in V(P)$, and v_0 is not a middle vertex.*

Proof. Suppose there exists a separating 2-chord $Q = v_0v_1v_2 \subset G$ where $|L(v_2)| = 3$, $v_0 \in V(P)$, and v_0 is not a middle vertex. Let G_1 and G_2 be the Q-components of G where G_1 contains the vertices of P.

By the minimality of G, there exists an L-coloring φ of G_1. Let L' be the list assignment on G_2 such that $L'(v_i) = \{\varphi(v_i)\}$ for $i \in \{0, 1, 2\}$ and $L'(x) = L(x)$ for $x \in V(G_2) \setminus V(Q)$. The $L'1$-vertices in G_2 are exactly those in Q.

Since all $L'2$-vertices in G_2 are also $L2$-vertices, we must verify that every $L'2$-vertex in G_2 has at most one neighbor in Q. If an $L'2$-vertex u has two neighbors, then one of them must be v_1 since G is C_4-free. However, at least one of the 2-chords v_0v_1u or v_2v_1u is separating and contradicts Claim 5.4.

Hence there exists an L'-coloring ψ of G_2 which agrees with φ on Q and together these colorings form an L-coloring of G. □

Our investigation of chords and 2-chords is complete. We now investigate the lists of adjacent vertices along the outer face in Claims 5.6 and 5.7.

Claim 5.6. *If $v_0v_1v_2$ is a path in F where $|L(v_1)| = 2$, then $L(v_1) \cap L(v_0) \neq L(v_1) \cap L(v_2)$.*

Proof. Suppose that there exists a path $v_0v_1v_2$ in F where $L(v_1) = \{a, b\}$ and $L(v_1) \cap L(v_0) = L(v_1) \cap L(v_2) = \{a\}$. We will find an L-coloring of G where v_1 is assigned the color b.

Let L' be the list assignment on $G - v_1$ where $L'(u) = L(u) \setminus \{b\}$ if $uv_1 \in E(G)$ and $L'(u) = L(u)$ otherwise. Let G' be obtained from $G - v_1$ by removing edges between $L'2$-vertices with disjoint lists. We will verify that G' and L' satisfy the hypothesis.

If u is a neighbor of v_1 with $b \in L(u)$, then u is not in F since by Claim 5.2 G contains no chord uv_1. Hence, the vertices that had the color b removed are now $L'2$-vertices, all $L'2$-vertices are on the outer face of G', and the $L'1$-vertices are exactly the vertices in P.

It remains to show that the $L'2$-vertices are independent in G' and no $L'2$-vertex has two neighbors in P. The $L2$-vertices in G still form an independent set in G'. The $L'2$-vertices that are neighbors of v_1 form an independent set since their L'-lists are pairwise disjoint (their L-lists previously contained b and cannot share more colors). If an $L2$-vertex u is adjacent to an $L'2$-vertex x that is a neighbor of v_1, then since $u \notin \{v_0, v_2\}$, the path uxv_1 is a separating 2-chord contradicting Claim 5.4. Similarly, if a neighbor x of v_1 is adjacent to two vertices u_0, u_1 of P, then at least one of them, say u_1, is not a middle vertex, and when $u_1 \notin \{v_0, v_2\}$ the path v_1xu_1 is a separating 2-chord contradicting Claim 5.4. If $u_1 \in \{v_0, v_2\}$, then since G contains no 4-cycles, the vertices u_0 and u_1 are adjacent and $v_1xu_0u_1$ is a 4-cycle.

Thus the hypothesis holds on G' and L', so by the minimality of G there exists an L'-coloring φ of G' which extends to an L-coloring of G with $\varphi(v_1) = b$. □

Claim 5.7. *If $v_0v_1v_2$ is a path in F where $|L(v_1)| = 3$, then v_0 and v_2 are $L2$-vertices, and the only $L2$-vertices adjacent to v_1.*

Proof. For a path $v_0v_1v_2$ where $|L(v_1)| = 3$, we consider how many of v_0 and v_2 are $L2$-vertices.

Suppose that neither v_0 nor v_2 is an $L2$-vertex. By Claim 5.2, G contains no good chord, and G contains no bad chord v_1u since v_0 and v_2 are not $L2$-vertices. Thus, all neighbors of v_1 other than v_0 and v_2 are $L3$-vertices. Select a color $a \in L(v_1)$ and let L' be the list assignment on G where $L'(z) = L(z)$ for $z \in V(G) \setminus \{v_1\}$ and $L'(v_1) = L(v_1) \setminus \{a\}$. If v_1 is adjacent to two vertices of P, they are v_0 and v_2, and $F = P \cup \{v_1\}$. This contradicts that $|V(F)| \geq 5$. Thus, the hypothesis holds on G with lists L' and by the minimality of G guarantees an L'-coloring of G, which is an L-coloring of G.

Now suppose that v_2 is an $L2$-vertex and v_0 is not. By Claim 5.2, G contains no good chord, and if G contains a bad chord v_1u it is with a triangle v_1uv_2, and we can write $u = v_3$ as the other neighbor of v_2 on F; in this case, v_3 is an $L3$-vertex since it is adjacent to v_2. Thus, all neighbors of v_1 other than v_0 and v_2 are $L3$-vertices.

Let a be the color in $L(v_1) \cap L(v_2)$. Let G' be obtained from G by removing the edge v_1v_2 and L' be the list assignment where $L'(v_1) = L(v_1) \setminus \{a\}$ and $L'(x) = L(x)$ for $x \in V(G) \setminus \{v_1\}$. Since the only $L2$-vertex adjacent in G to v_1 is v_2, and they are not adjacent in G', the $L'2$-vertices form an independent set in G'. Moreover, Claim 5.3 implies that v_1 has at most one neighbor in P. Hence G' satisfies the hypothesis, and by the minimality of G there exists an L'-coloring of G'. By the construction of L' and G', φ is also an L-coloring of G.

Thus, for a path $v_0v_1v_2$ in F with v_1 an $L3$-vertex, v_0 and v_1 are both $L2$-vertices. Since every bad chord v_1u has u adjacent to v_0 or v_2, the vertex u is an $L3$-vertex. Thus Claim 5.2 implies that v_0 and v_2 are the only $L2$-vertices adjacent to v_1. □

By the minimality of the sum of the sizes of the lists, we can assume that $|V(P)| \geq 1$ by removing colors if necessary. Let $p_0v_1v_2v_3 \dots v_tv_{t+1} \dots$ be vertices of F in cyclic order where $p_0 \in V(P)$, $\{v_1, \dots, v_t\} = V(F) \setminus V(P)$, and thus $v_{t+1} \in V(P)$.

Claims 5.6 and 5.7 together imply that for all $i \in \{1, \dots, t\}$, the vertex v_i is an $L2$-vertex when i is odd; otherwise v_i is an $L3$-vertex. Furthermore, v_t is an $L2$-vertex, so t is odd.

Select a set $X \subseteq \{v_2, v_3, v_4\}$ and a partial L-coloring φ of X by the following rules:

(X1) If v_2v_4 is not a bad chord, then let $c \in L(v_2) \setminus (L(v_1) \cup L(v_3))$ and:

- (X1a) If there is no common neighbor w of v_2 and v_3 such that $c \in L(v_2) \cap L(w)$, then let $X = \{v_2\}$ and $\varphi(v_2) = c$.
- (X1b) If there is a common neighbor w of v_2 and v_3 such that $c \in L(v_2) \cap L(w)$, then let $X = \{v_2, v_3\}$, $\varphi(v_2) = c$, and $\varphi(v_3) = b$ where b is the unique color in $L(v_3) \setminus L(v_4)$.

(X2) If v_2v_4 is a bad chord, then let $X = \{v_2, v_3, v_4\}$. If v_4 and v_5 have a common neighbor w, then let $\varphi(v_4) \in L(v_4) \setminus (L(v_5) \cup L(w))$; otherwise let $\varphi(v_4) \in L(v_4) \setminus L(v_5)$. Finally, select $\varphi(v_2) \in L(v_2) \setminus (L(v_1) \cup \{\varphi(v_4)\})$ and $\varphi(v_3) \in L(v_3) \setminus \{\varphi(v_4)\}$ such that $\varphi(v_2) \neq \varphi(v_3)$

Observe that X and φ are well-defined, since there is always a choice for φ satisfying those rules. See Figure 1 for diagrams of these cases.

Let L' be a list assignment on $G - X$ where

$$L'(v) = L(v) \setminus \{\varphi(x) : x \in X \text{ and } xv \in E(G)\}$$

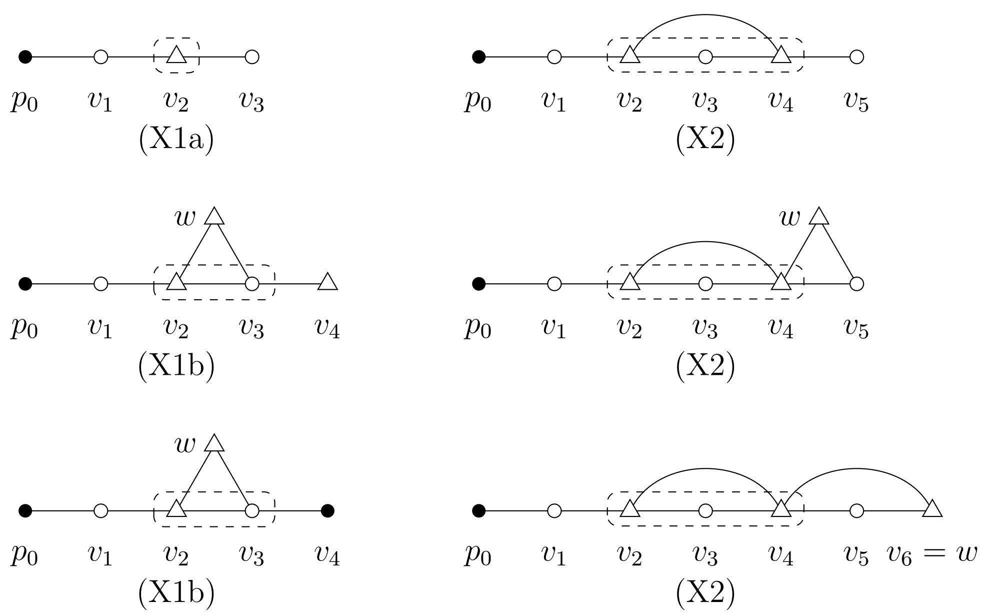

Figure 1: Cases (X1) and (X2). A black circle is an $L1$-vertex, a white circle is an $L2$-vertex, and a triangle is an $L3$-vertex. The dashed box indicates X.

for all $v \in V(G) \setminus X$. Let G' be obtained from $G - X$ by removing edges among vertices with disjoint L'-lists except the edges of P.

Below, we verify that G', L', and P satisfy the assumptions of Theorem 5. Then by the minimality of G, there is an L'-coloring ψ of G'. By the definition of L', the colorings φ and ψ together form an L-coloring of G, a contradiction.

Let N be the set of vertices u where $|L(u)| > |L'(u)|$. Necessarily, every vertex of N has a neighbor in X. Observe that X and φ are chosen such that $L(u) = L'(u)$ for all $u \in V(F) \setminus X$. Hence $N \subseteq V(G) \setminus V(F)$ and every vertex in N is an $L3$-vertex.

Since G is C_4-free, any pair of vertices has at most one common neighbor. When $|X| = 3$, we are in the case (X2), and the chord v_2v_4 implies that no vertex in N is adjacent to v_3, and a vertex adjacent to v_2 and v_4 would form a 4-cycle with v_3. When $|X| = 2$, there is at most one vertex in N having two neighbors in X. This is possible only in the case (X1b), and the colors $\varphi(v_2)$ and $\varphi(v_3)$ are chosen so that the common neighbor is an $L'2$- or $L'3$-vertex. Therefore $|L'(v)| \geq 2$ for every vertex $u \in N$.

If two vertices $x, y \in N$ are adjacent in G', the color $c \in L(x) \cap L(y)$ is also in $L'(x) \cap L'(y)$ and hence the colors $a \in L(x) \setminus L'(x)$ and $b \in L(y) \setminus L'(y)$ are distinct. Thus, x is adjacent to some $v_i \in X$ where $\varphi(v_i) = a$, and y is adjacent to some $v_j \in X$ where $\varphi(v_j) = b$. In every case above, any two distinct vertices in X that have neighbors not in X are also adjacent, so xv_iv_jy is a 4-cycle. Thus, N is an independent set.

Suppose that there is an edge $uv \in E(G')$ where $u \in N$ and $v \in V(F) \setminus X$ where $|L'(v)| = |L(v)| = 2$. If the 2-chord xuv is separating, we find a contradiction by Claim 5.4.

If the 2-chord is not separating, then x and v are consecutive in F, and exactly one is in X.

First, we consider the case when $xuv = v_2uv_1$. If $L(v_1) \cap L(u) = L(p_0)$, then the edge v_1u does not restrict the colors assigned to v_1 and u by an L-coloring, so G is not minimal; thus $L(v_1) \cap L(u) \neq L(p_0)$. Hence the vertices v_1, u, and v_2 all share a common color, and this color was not removed from the list $L(u)$, so $|L'(u)| = 3$.

When $xuv \neq v_2uv_1$, then $xuv = v_iuv_{i+1}$, where i is maximum such that $v_i \in X$. However, the cases (X1a), (X1b), and (X2) all consider whether v_i and v_{i+1} have a common neighbor, and avoid using any color in common if v_{i+1} is an $L2$-vertex. Therefore, u is an $L'3$-vertex, so the $L'2$-vertices in G' form an independent set.

Finally, we verify that no $L'2$-vertex in G' has two neighbors in P. Since G' is obtained from G by deletions of edges and vertices, it suffices to check the condition only for vertices in N. If $v \in N$ has a neighbor $x \in X$, then v is not adjacent to two vertices of P by Claims 5.4 and 5.5 and G being C_4-free.

Therefore, G', L', and P satisfy the assumptions of Theorem 5. □

4 Forbidding 5- and 6-cycles

The goal of this section is to prove Theorem 3. We prove a slightly stronger statement.

Theorem 6. *Let G be a plane graph without 5- or 6-cycles and let $p \in V(G)$. Let L be a $(*, 1)$-list assignment such that*

- $|L(p)| = 1$,
- $|L(v)| = 3$ *for* $v \in V(G) - p$.

Then G is L-colorable.

This strengthening allows us to assume that a minimum counterexample is 2-connected, since we can iteratively color a graph by its blocks using at most one precolored vertex at each step.

Our proof uses a discharging technique. In Section 4.1, we define a family of *prime graphs* and prove in Section 4.3 that a minimum counterexample is prime. The proof is then completed in Section 4.2, where we define a discharging process and prove that prime graphs do not exist, and hence a minimum counterexample does not exist.

4.1 Configurations

We introduce some notation for a plane graph G. Let $V(G)$, $E(G)$, and $F(G)$ be the set of vertices, edges, and faces, respectively. For $v \in V(G)$, let $d(v) = |N(v)|$ where $N(v)$ is the set of vertices adjacent to v. For $f \in F(G)$, let $d(f)$ be the length of f.

For a C_5- and C_6-free plane graph, the subgraph of the dual graph induced by the 3-faces has no component with more than three vertices. A *facial* K_4 is a set of three pairwise adjacent 3-faces. We say four vertices xz_1yz_2 form a *diamond* if xz_1yz_2 is a 4-cycle formed by two adjacent 3-faces xyz_i for $i \in \{1, 2\}$. If a 3-face is not adjacent to another 3-face, then it is *isolated*.

A vertex is *low* if it has degree three; otherwise it is *high*. A 3-face is *bad* if it is incident to a low vertex; otherwise it is *good*. A face is *small* if it has length three or four. A face is *large* if it has length at least seven. A 4-face is *special* if is is incident to p and *normal* otherwise.

Definition 7. For a plane graph G, a list assignment L from Theorem 6, and $p \in V(G)$ with $|L(p)| = 1$, the pair (G, L) is *prime* if

- G is 2-connected
- $d(v) \geq 3$ for every $v \in V(G) - p$
- $d(p) \geq 2$

and in addition (G, L) contains none of the configurations (C1)–(C16) below:

(C1) A 3-face containing p.

(C2) A normal 4-face where all incident vertices are low.

(C3) A 3-face incident to at most one high vertex.

(C4) A diamond xz_1yz_2 where $d(z_1) = d(z_2) = 3$ and $d(x) = d(y) = 4$.

(C5) A diamond xz_1yz_2 where $d(y) = 4$ and $d(x) = 3$.

(C6) A facial K_4 $wxyz$ where w is the internal vertex, and at least one of x, y, and z has degree at most 4.

(C7) A facial K_4 $wxyz$ where w is the internal vertex, the vertex z has degree exactly five, and the other two neighbors u, v of z bound a bad 3-face zvu.

(C8) A diamond xz_1yz_2 where $d(z_1) = d(z_2) = 3$, $d(x) = 4$, $d(y) = 5$, and the other two neighbors u, v of y form a bad 3-face yvu.

(C9) A diamond xz_1yz_2 where $d(x) = 3$, $d(y) = 5$, and the other two neighbors u, v of y form a bad 3-face yvu.

(C10) A diamond xz_1yz_2 where $d(z_2) = 3$, $d(x) = d(y) = d(z_1) = 4$, and the other two neighbors u, v of z_1 form a bad 3-face z_1vu.

(C11) A bad 3-face xyz and a normal 4-face $wuvx$ where $d(x) = 4$ and x is the only high vertex incident to the 4-face.

(C12) Two 3-faces xyz and xuv where $d(x) = 4$ and $d(y) = d(v) = 3$.

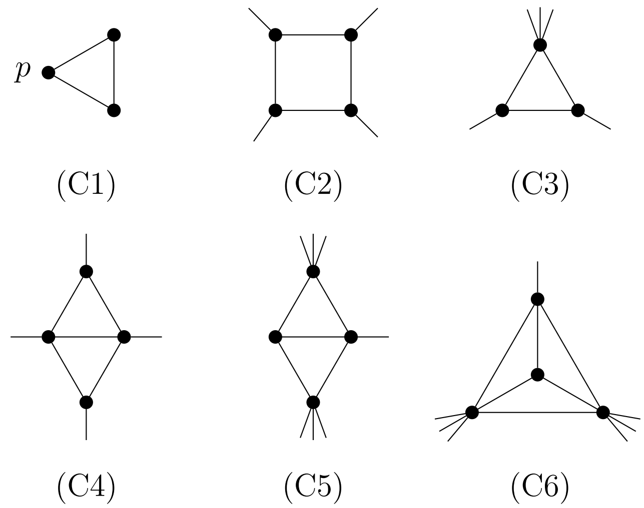

Figure 2: Simple reducible configurations.

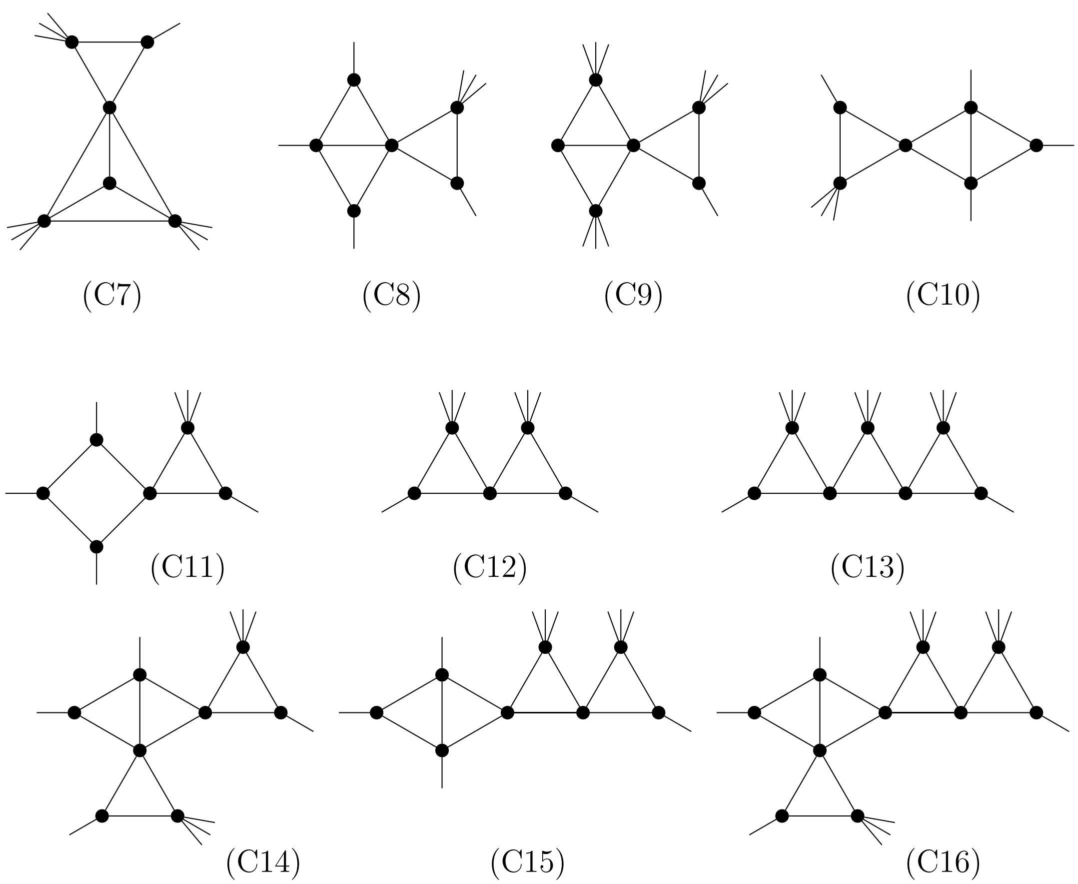

Figure 3: Compound reducible configurations.

(C13) Three 3-faces xyz, xuv, vpq, where $d(y) = d(p) = 3$ and $d(x) = d(v) = 4$.

(C14) A diamond xz_1yz_2 where xy is an edge, $d(z_1) = 3$, $d(y) = 4$, $d(x) = 5$, and $d(z_2) = 4$, where x and z_2 are each incident to a bad 3-face.

(C15) A diamond xz_1yz_2 where xy is an edge, $d(z_1) = 3$ and $d(y) = d(x) = d(z_2) = 4$, where z_2 is incident to a good 3-face z_2uv with $d(v) = 4$ and v is incident to another bad 3-face.

(C16) A diamond xz_1yz_2 where xy is an edge, $d(z_1) = 3$, $d(x) = 5$, and $d(y) = d(z_2) = 4$, where x is incident to a bad 3-face and z_2 is incident to a good 3-face z_2uv with $d(v) = 4$ and v is incident to another bad 3-face.

The configurations (C1)–(C6) are called *simple*. See Figure 2. Other configurations can be built from simple ones by replacing an edge with one endpoint in the configuration by a bad 3-face; we call these *compound*. See Figure 8 for a sketch of creating compound configurations. For convenience, we list compound configurations used in our proof. See Figure 3. Reducibility is proved in Lemma 19 from Section 4.3.

Observe that a prime graph G has no 5- or 6-faces since no 5- or 6-cycles exist and G is 2-connected.

4.2 Discharging

In this section, we prove the following proposition.

Proposition 8. *No pair (G, L) is prime.*

We shall prove that a prime (G, L) does not exist by assigning an initial *charge* $\mu(z)$ to each $z \in V(G) \cup F(G)$ with strictly negative total sum, then applying a discharging process to end up with charge $\mu^*(z)$. We prove that since (G, L) does not contain any configuration in (C1)–(C16), then μ^* has nonnegative total sum. The discharging process will preserve the total charge sum, and hence we find a contradiction and G does not exist.

For every vertex $v \in V(G) - p$ let $\mu(v) = 2d(v) - 6$, for p let $\mu(p) = 2d(p)$, and for every face $f \in F(G)$, let $\mu(f) = d(f) - 6$. The total initial charge is negative by

$$\begin{aligned}\sum_{z \in V(G) \cup F(G)} \mu(z) &= \sum_{v \in V(G)-p} (2d(v) - 6) + 2d(p) + \sum_{f \in V(F)} (d(f) - 6) \\ &= 6|E(G)| - 6|V(G)| - 6|F(G)| + 6 = -6.\end{aligned}$$

The final equality holds by Euler's formula.

In the rest of this section we will prove that the sum of the final charge after the discharging phase is nonnegative. Instead of looking at each individual face, we look at groups of adjacent 3-faces.

Note that since G has no 5-cycles and 6-cycles, no 4-face is adjacent to a 3- or 4-face (and hence every face adjacent to a 4-face has length at least seven). If a vertex v with $d(v) \geq 4$ is

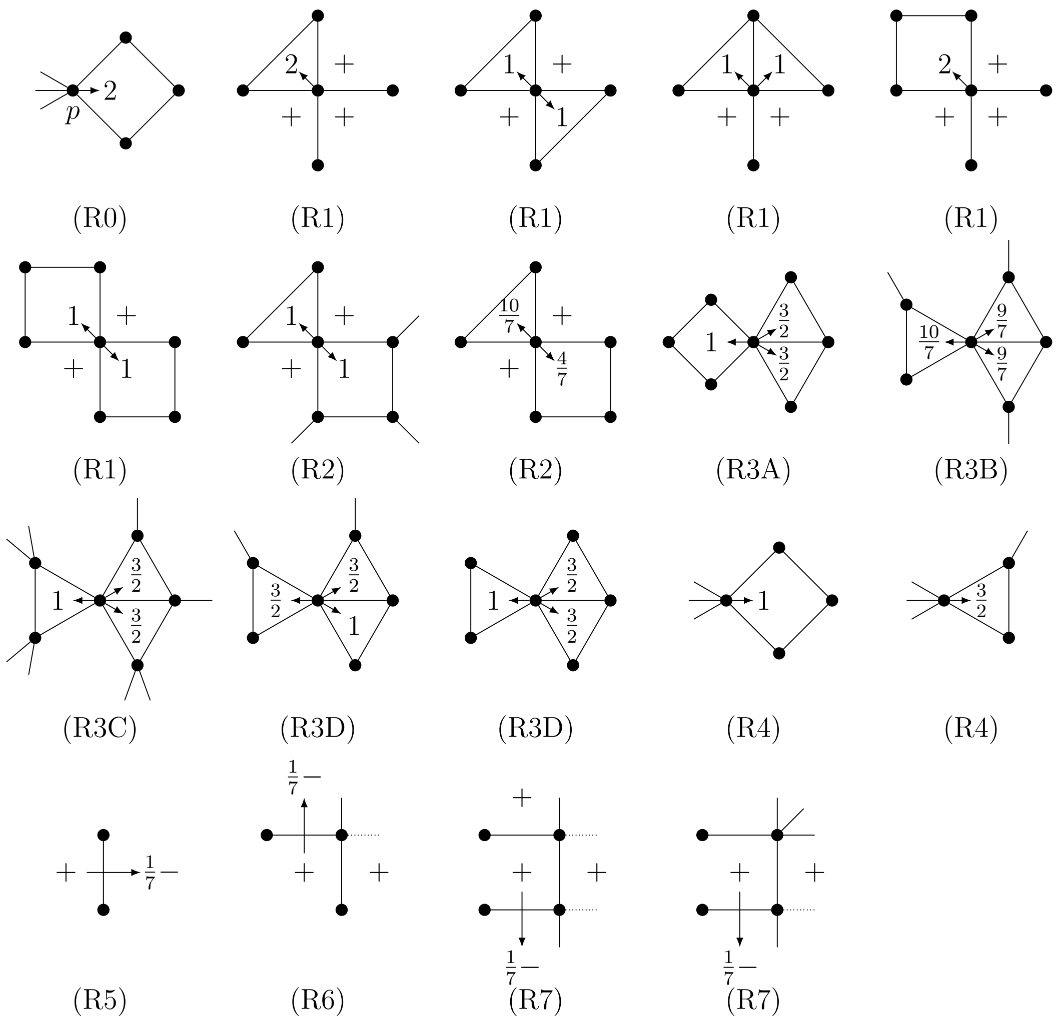

Figure 4: Discharging rules.

incident to ℓ_3 3-faces and ℓ_4 4-faces, then $d(v) \geq \frac{3}{2}\ell_3 + 2\ell_4$. Thus, every vertex v is incident to at most $2d(v)/3$ small faces.

We begin by discharging from vertices with positive charge to small faces with negative charge. The precolored vertex p transfers charge according to rule (R0).

(R0) p sends charge 2 to every (special) incident 4-face.

For a vertex $v \in V(G) - p$ with $d(v) \geq 4$, exactly one of the discharging rules (R1)–(R4) applies; rules (R0)–(R4) are called *vertex rules.*

(R1) If $d(v) = 4$ and v is not incident to both a 3-face and a normal 4-face, then v distributes its charge uniformly to each incident 3-face or normal 4-face.

(R2) If $d(v) = 4$ and v is incident to a 3-face t and a normal 4-face f, then:

- (R2A) If f is incident to exactly one high vertex, then v gives charge 1 to f and 1 to t.
- (R2B) If f is incident to more than one high vertex, then v gives charge $\frac{4}{7}$ to f and $\frac{10}{7}$ to t.

(R3) If $d(v) = 5$, then:

- (R3A) If v is incident to a normal 4-face, then v gives charge 1 to each normal 4-face and distributes its remaining charge uniformly to each incident 3-face.
- (R3B) If v is incident to three bad 3-faces, then v gives charge $\frac{10}{7}$ to the isolated 3-face and $\frac{9}{7}$ to each 3-face in the diamond.
- (R3C) If v is incident to only one bad 3-face that is in a diamond with another 3-face incident to v, then v gives charge $\frac{3}{2}$ to both 3-faces in the diamond, and if v is incident to another 3-face t, then v gives charge 1 to t.
- (R3D) Otherwise, v gives charge $\frac{3}{2}$ to each incident bad 3-face and distributes its remaining charge uniformly to each incident non-bad 3-face.

(R4) If $d(v) \geq 6$, then v gives charge 1 to each incident normal 4-face and charge $\frac{3}{2}$ to each incident 3-face.

After applying the vertex rules, we say a face is *hungry* if it is a negatively-charged small face, or it is a 3-face in a negatively-charged diamond.

We now discharge from large faces to hungry faces; the rules (R5)–(R7) are *face rules.* Let f be a face with $d(f) \geq 7$ and let $f_0, f_1, f_2, \dots, f_{d(f)} = f_0$ be the faces adjacent to f in counterclockwise order. Observe that f has charge at least $d(f)/7$, and so f could send charge $\frac{1}{7}$ to each adjacent face. Each of the rules below could apply to f and an adjacent face f_i, to decide where the charge $\frac{1}{7}$ associated with f_i should go.

(R5) If f_i is hungry, then f gives charge $\frac{1}{7}$ to f_i.

(R6) If f_i is not hungry, f_{i+1} is hungry, and the vertex incident to f, f_i, and f_{i+1} has degree at most four, then f gives charge $\frac{1}{7}$ to f_{i+1} instead of f_i.

(R7) If f_i is not hungry, f_{i-1} is hungry, the vertex incident to f, f_{i-1}, and f_i has degree at most four, and either the vertex incident to f, f_i, and f_{i+1} has degree at least five or f_{i+1} is not hungry, then f gives charge $\frac{1}{7}$ to f_{i-1} instead of f_i.

We now show that the discharging rules result in a nonnegative charge sum $\sum_{v \in V(G)} \mu^*(v) + \sum_{f \in F(G)} \mu^*(f) \geq 0$, contradicting our previously computed sum of -6. First, we prove that the final charge μ^* is nonnegative on every vertex. Then, we prove that the final charge μ^* is nonnegative on every large face and every 4-face. A set S of 3-faces is *connected* if they induce a connected subgraph of the dual graph. Since G contains no 5- or 6-cycles, a connected set of 3-faces is either a facial K_4, a diamond, or an isolated 3-face. We will show that for every connected set S of 3-faces, the final charge sum $\sum_{f \in S} \mu^*(f)$ is nonnegative.

Claim 8.1. *For each vertex $v \in V(G)$, the final charge $\mu^*(v)$ is nonnegative.*

Proof. If $v = p$, then rule (R0) applies then p is incident to at most $d(p)/2$ 4-faces since 4-faces cannot share an edge. So $\mu^*(v) \geq 2d(p) - 2d(p)/2 \geq 0$.

Assume $v \in V(G) \setminus p$. Recall $d(v) \geq 3$. If $d(v) = 3$, then $\mu(v) = 0$ and no charge is sent from this vertex.

If $d(v) = 4$, then $\mu(v) = 2$ and (R1) or (R2) applies. Consider the four faces incident to v. Since G avoids 5- and 6-cycles, at most two of these faces are small. If v is not incident to both a 3- and 4-face, then (R1) applies and v sends all charge uniformly to each small face; hence $\mu^*(v) = 0$. If v is incident to both a 3- and 4-face, then (R2) applies and v sends total charge two to the two small faces (either as $1 + 1$ for (R2A) or $\frac{4}{7} + \frac{10}{7}$ for (R2B)).

If $d(v) = 5$, then $\mu(v) = 4$ and (R3) applies. There are five faces incident to v, and since G avoids 5- and 6-cycles, at most three of these faces are small. If v is incident to three bad 3-faces, then two of the faces are adjacent so these faces partition into a bad 3-face and a diamond; (R3B) applies and a total charge of four is sent from v, so $\mu^*(v) = 0$. If v is not incident to three bad 3-faces, v is incident to at most two 3- or 4-faces; (R3A), (R3C), or (R3D) applies, and v sends at most charge three, $\mu^*(v) \geq 0$.

If $d(v) \geq 6$, then (R4) applies. Since G avoids 5- and 6-cycles, v is incident to at most $\frac{2d(v)}{3}$ small faces. Since $\mu(v) = 2d(v) - 6$ and v sends charge at most $\frac{2d(v)}{3} \cdot \frac{3}{2} = d(v)$, the final charge on v is $\mu^*(v) \geq d(v) - 6 \geq 0$. □

Claim 8.2. *For each face $f \in F(G)$ with $d(f) \geq 7$, the final charge $\mu^*(f)$ is nonnegative.*

Proof. Let the faces adjacent to f be listed in clockwise order as $f_1, f_2, \ldots$ as in the discharging rules. Observe that each adjacent face f_i satisfies at most one of the rules (R5), (R6), and (R7), and hence f sends charge $\frac{1}{7}$ at most $d(f)$ times, leaving $\mu^*(f) \geq \mu(f) - \frac{d(f)}{7} \geq 0$. □

Claim 8.3. *For each 4-face $f \in F(G)$, the final charge $\mu^*(f)$ is nonnegative.*

Proof. If f is a special 4-face, (R0) applies. Thus $\mu^*(f) = -2 + 2 = 0$. So assume that f is a normal 4-face.

Observe $\mu(f) = -2$, and all faces adjacent to f have length at least seven. Since G contains no (C2), the normal 4-face f is incident to at least one high vertex.

If f is incident to exactly one high vertex v, then v sends charge at least 1 to f by (R1), (R2A), or (R4). The four incident faces each send charge at least $\frac{1}{7}$ by (R5). Three of the four faces adjacent to f each send an additional $\frac{1}{7}$ by (R6). Thus, $\mu^*(f) \geq -2+1+4\cdot\frac{1}{7}+3\cdot\frac{1}{7} = 0$.

If f is incident to exactly two high vertices u and v, then u and v each send charge at least $\frac{4}{7}$ by (R1), (R2B), (R3A), or (R4). The four incident faces each send charge $\frac{1}{7}$ by (R5). Two of the four faces adjacent to f each send an additional $\frac{1}{7}$ by (R6). Thus, $\mu^*(f) \geq -2 + 2\cdot\frac{4}{7} + 4\cdot\frac{1}{7} + 2\cdot\frac{1}{7} = 0$.

If f is incident to at least three high vertices, then each high vertex sends charge at least $\frac{4}{7}$ by (R1), (R2B), (R3A), or (R4). The four faces adjacent to f each send charge at least $\frac{1}{7}$ by (R5). Thus, $\mu^*(f) \geq -2 + 3\cdot\frac{4}{7} + 4\cdot\frac{1}{7} \geq 0$. □

We now show the total charge sum over the 3-faces is nonnegative by showing the charge sum is nonnegative on each connected set of 3-faces, starting with facial K_4's (Claim 8.4), then diamonds (Claim 8.5), and finally isolated 3-faces (Claim 8.6).

Observe that if t is a good 3-face, then t receives charge at least 1 from every incident vertex by the vertex rules.

Claim 8.4. *For each facial K_4, the sum of the final charge of the three 3-faces is nonnegative.*

Proof. Let $wxyz$ be a facial K_4 where w is the internal vertex. Since G contains no (C6), all vertices $v \in \{x, y, z\}$ have degree $d(v) \geq 5$. Since G contains no (C7), any vertex $v \in \{x, y, z\}$ with $d(v) = 5$ is not incident to another bad 3-face outside the facial K_4. Thus, each vertex $v \in \{x, y, z\}$ sends charge at least $2\cdot\frac{3}{2}$ by (R3D) or (R4) to the 3-faces in the facial K_4, and $\mu^*(wxy) + \mu^*(wyz) + \mu^*(wzx) \geq -9 + 3 + 3 + 3 = 0$. □

Claim 8.5. *For each diamond, the sum of the final charge of the two 3-faces is nonnegative.*

Proof. Let the diamond have vertices xz_1yz_2 where xyz_i is a 3-face f_i for $i \in \{1, 2\}$. We assume $d(x) \leq d(y)$. Note that if the diamond does not have nonnegative charge after the vertex rules, then both faces f_1 and f_2 are hungry.

By our earlier observation, if both faces f_1 and f_2 are good, then they each receive charge at least 3 from the incident vertices, and the diamond has nonnegative charge after the vertex rules. We now consider which faces in f_1 and f_2 are bad.

Case 1: Exactly one 3-face f_i is bad. In this case, we will assume $d(z_1) \leq d(z_2)$, so it must be f_1 that is bad, while f_2 is good. Thus, $d(y) \geq d(x) \geq 4$, and $d(z_2) \geq 4$. Since f_1 is bad, $d(z_1) = 3$. For $v \in \{x, y\}$, let f_v be the face incident to v that follows f_1 and f_2 in the counterclockwise order around v.

If $d(y) \geq 5$ then y contributes at least $\frac{3}{2} + 1$ by (R3D), $2\cdot\frac{9}{7}$ by (R3B), or at least $2\cdot\frac{3}{2}$ by (R3A), (R3C), or (R4). If $d(y) = 4$ then y contributes at least 2 by (R1), but now (R6) also applies to the face f_y, adding an extra contribution of $\frac{1}{7}$. The contribution to the diamond is at least $2 + \frac{1}{7}$ from the vertex rules applied to y and (R6) applied to the face f_y. By symmetry, the contribution to the diamond is also at least $2 + \frac{1}{7}$ from the vertex rules applied to x and (R6) applied to the face f_x. By the vertex rules, z_2 sends charge at least 1 to f_2. Rule (R5), the four faces adjacent to the diamond send $\frac{1}{7}$ each. Rule (R6) applies to the face incident to z_1 that is not f_1, f_x, or f_y, giving $\frac{1}{7}$ to the diamond. Thus $\mu^*(f_1) + \mu^*(f_2) \geq -6 + 2(2 + \frac{1}{7}) + 1 + 4 \cdot \frac{1}{7} + \frac{1}{7} \geq 0$.

Case 2: Both 3-faces f_1 and f_2 are bad. We consider the degree of x and order z_1 and z_2 such that z_1, y, and z_2 appear consecutively in the clockwise ordering of the neighbors of x. Observe that when $d(x) > 3$, we have $d(z_1) = d(z_2) = 3$ since f_1 and f_2 are bad.

Case 2.i: $d(x) \geq 5$. By (R3) and (R4), both x and y each send at least $2 \cdot \frac{9}{7}$. By (R5), the four incident faces contribute charge $4 \cdot \frac{1}{7}$ to the diamond and by (R6), two of the faces incident to z_1 or z_2 contribute at least $2 \cdot \frac{1}{7}$. Thus the final charge on the diamond is $\mu^*(f_1) + \mu^*(f_2) \geq -6 + 4 \cdot \frac{9}{7} + 4 \cdot \frac{1}{7} + 2 \cdot \frac{1}{7} = 0$.

Case 2.ii: $d(x) = 4$. By (R1), the vertex x sends charge 1 to each face f_i. Since G contains no (C8), if $d(y) = 5$ then y is not incident to a bad 3-face other than f_1 and f_2. Thus, y sends charge $\frac{3}{2}$ to each face f_i, and after the vertex rules the charge on the diamond is $-6 + 2 \cdot 1 + 2 \cdot \frac{3}{2} = -1$, and the faces f_i are hungry. By (R5), the four faces adjacent to f_1 and f_2 each send charge $\frac{1}{7}$ to the diamond. By (R6), three of the four faces adjacent to f_1 and f_2 each send charge $\frac{1}{7}$ to the diamond. Thus, $\mu^*(T_1) + \mu^*(T_2) = -6 + 2 \cdot 1 + 2 \cdot \frac{3}{2} + 4 \cdot \frac{1}{7} + 3 \cdot \frac{1}{7} \geq 0$.

Case 2.iii: $d(x) = 3$. Since G contains no (C5), we have $d(y) \geq 5$. Let f be the face incident to z_1, x, z_2. If f is a 3-face, then $z_1 x z_2 y$ is a facial K_4, handled in Claim 8.4. If f is a 4-face, then G contains a 5-cycle, a contradiction. Thus, $d(f) \geq 7$, and let $s_0, s_1, s_2, \ldots$ be the faces adjacent to f in cyclic clockwise order where $s_1 = f_1$ and $s_2 = f_2$. Since G contains no (C9), $d(y) = 5$ implies that the vertex y is not adjacent to a bad 3-face other than f_1 and f_2. By (R3C), (R3D), and (R4), y sends charge $\frac{3}{2}$ to each face f_i. By (R5), four faces adjacent to the diamond each contribute at least $\frac{1}{7}$. Since G contains no (C3), it follows that $d(z_i) \geq 4$ for each $i \in \{1, 2\}$.

If $d(z_i) \geq 5$ for some $i \in \{1, 2\}$, then by the vertex rules, z_1, z_2 together will contribute at least $1 + \frac{10}{7}$ to the diamond. Thus, $\mu^*(f_1) + \mu^*(f_2) \geq -6 + 1 + \frac{10}{7} + 3 + 4 \cdot \frac{1}{7} \geq 0$. We can now assume $d(z_1) = d(z_2) = 4$.

(a) If $d(s_3) \geq 7$, then z_2 sends charge 2 to f_2 by (R1) and z_1 sends charge at least 1 to f_1 by the vertex rules. Thus $\mu^*(f_1) + \mu^*(f_2) \geq -6 + 2 \cdot \frac{3}{2} + 2 + 1 \geq 0$. By symmetry this also solves the case when $d(s_0) \geq 7$.

(b) If $d(s_3) = 4$, then since s_3 and s_2 are not a copy of (C11), the face s_3 must be incident to at least two high vertices. By (R2B), z_3 sends charge $\frac{10}{7}$ to f_2 and by the vertex rules, z_1 sends charge at least 1 to f_2. Thus, $\mu^*(f_1) + \mu^*(f_2) \geq -6 + 2 \cdot \frac{3}{2} + \frac{10}{7} + 1 + 4 \cdot \frac{1}{7} = 0$. By symmetry this also solves the case when $d(s_0) = 4$.

(c) If $d(s_0) = d(s_3) = 3$, then since the faces s_0 and f_1 (or the faces f_2 and s_3) do not form

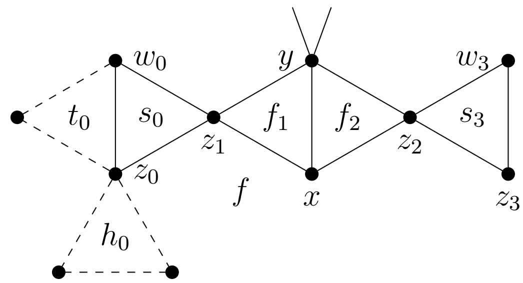

Figure 5: Situation in Case 3.iii(c).

a copy of (C12), the 3-faces s_0 and s_3 are not bad 3-faces.

Let z_0 be the vertex such that z_0z_1 is the edge between f and s_0; similarly let z_3 be the vertex such that z_3z_2 is the edge between f and s_3. Let w_i be the other vertex of s_i different from z_i for $i \in \{0, 3\}$. For $i \in \{0, 3\}$, we have $d(z_i) \geq 4$ and $d(w_i) \geq 4$ since s_i is not a bad 3-face. See Figure 5 for a sketch of the situation.

Let i be in $\{0, 3\}$. If s_i is an isolated 3-face, then it receives charge at least 1 from each of its incident vertices and is not hungry after the vertex rules. If s_i is in a diamond with a bad 3-face t_i then since $t_i, s_i, f_{|i-1|}$ is not a copy of (C10), at least one of z_i and w_i has degree at least 5. By symmetry, assume z_i has degree at least 5. If (R3D) does not apply to z_i, then the diamond formed by s_i, t_i receives charge at least 6 from its vertices and is not hungry after the vertex rules. Hence (R3D) applies to z_i and there must be a bad 3-face h_i incident to z_i that is not t_i. If w_i has degree at least 5, by symmetry, (R3D) applies and t_i, s_i are not hungry. If w_i has degree 4 then the faces s_i, t_i, h_i, and $f_{|i-1|}$ form a copy of (C14). Therefore the diamond s_i, t_i is not hungry after the vertex rules. Therefore s_0 and s_3 are not hungry after the vertex rules.

By (R5), the three faces adjacent to f_1 and f_2 send charge $4 \cdot \frac{1}{7}$ to the diamond. The rule (R6) applied on f and edge z_2z_3 sends charge $\frac{1}{7}$ to f_2 and (R6) applied on edge z_1w_0 and face containing z_1w_0 sendscharge $\frac{1}{7}$ to f_1. Finally, we show that (R7) applies on z_1z_0 which gives additional $\frac{1}{7}$. If $d(z_0) \geq 5$ then (R7) applies. Suppose that $d(z_0) = 4$. If s_0 is in a diamond with t_0, then (R6) cannot apply on z_1z_0. If z_0 is in another bad 3-face h_0, then h_0, s_0, f_1 form reducible configuration (C13). Hence (R7) indeed applies. Thus, $\mu^*(f_1) + \mu^*(f_2) \geq -6 + 2 \cdot \frac{3}{2} + 1 + 1 + 4 \cdot \frac{1}{7} + 2 \cdot \frac{1}{7} + \frac{1}{7} \geq 0$.

In all cases, our diamond has nonnegative total charge. □

Claim 8.6. *For each isolated 3-face t, the final charge $\mu^*(t)$ is nonnegative.*

Proof. Note that $\mu(t) = -3$. If t is good, then each incident vertex sends charge at least 1 by the vertex rules and $\mu^*(t) \geq -3 + 3 = 0$. We now assume t is incident to at least one low vertex. Moreover, we assume that t is hungry.

Since G contains no (C3), t is incident to exactly one low vertex. Let x, y, and z be the vertices incident to t in counter-clockwise order where x is low, and let f be the face sharing

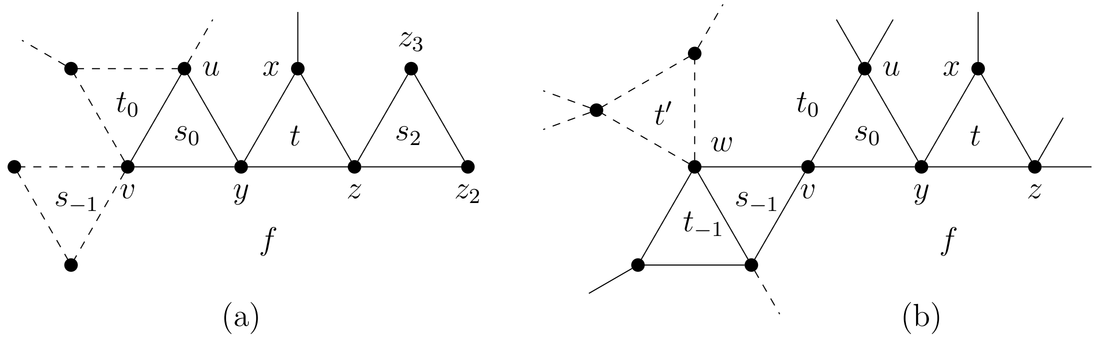

Figure 6: Claim 8.6 where $d(s_0) = d(s_2) = 3$.

the edge yz with t. The three faces adjacent to t each send charge $\frac{1}{7}$ to t by (R5). The faces having zx in common with t sends charge $\frac{1}{7}$ to t by (R6).

If one of y and z has degree at least 5, then y and z together send charge at least $1 + \frac{10}{7}$ to t by the vertex rules. Thus, $\mu^*(t) \geq -3 + 1 + \frac{10}{7} + 3 \cdot \frac{1}{7} + \frac{1}{7} \geq 0$.

We now assume $d(y) = d(z) = 4$. Let $s_0, s_1, s_2, \dots$ be the faces adjacent to f in clockwise order so that $s_1 = t$. Observe y is incident to s_0 and s_1, while z is incident to s_1 and s_2.

If $d(s_2) \geq 7$, then z sends charge 2 to t by (R1) and y sends charge at least 1 to t by the vertex rules, so $\mu^*(t) \geq -3 + 2 + 1 = 0$. Hence $d(s_2) \leq 4$ and by symmetry $d(s_1) \leq 4$.

If $d(s_2) = 4$, then since G contains no (C11), s_2 must be incident to at least two high vertices. Thus, z sends charge $\frac{10}{7}$ to t by (R2B), and y sends charge at least 1 to t by the vertex rules. Therefore, $\mu^*(t) \geq -3 + \frac{10}{7} + 1 + 3 \cdot \frac{1}{7} + \frac{1}{7} \geq 0$. Hence $d(s_2) = 3$ and by symmetry, also $d(s_0) = 3$.

Since neither the pair s_0 and s_1, nor the pair s_1 and s_2 form a copy of (C12), the faces s_0 and s_2 are good 3-faces. Let $V(s_0) = \{u, v, y\}$ so that vy is the edge between s_0 and f. See Figure 6, for an example of this situation.

Suppose that s_0 is hungry after vertex rules. Since s_0 is not a bad face, it must be in a diamond with a bad face t_0. Since (C10) is reducible, at least one of u, v has degree at least 5. If both u and v have degree at least 5 or one has degree at least 6, then t_0 and s_0 receive enough charge after the vertex rules and s_0 is not hungry. Hence without loss of generality, assume $d(v) = 5$ and $d(u) = 4$. If v is incident to another bad face s_{-1}, then s_0, s_{-1}, t_0, s_t form a reducible configuration (C14). Hence s_{-1} is not a bad 3-face, so (R3C) applies and the faces s_0 and t_0 are not hungry.

By a symmetric argument, s_2 is also not hungry after the vertex rules.

Recall that $d(x) = 3$, $d(y) = d(z) = 4$, and s_0 and s_2 are good 3-faces. Thus, y and z send charge 1 to t by (R1). By (R6), the three faces adjacent to t gives charge $\frac{3}{7}$ to t. If $d(v) \geq 5$, then (R7) applies to f and f contributes $\frac{1}{7}$ to t. Hence $d(v) = 4$. If s_{-1} is not hungry, then again (R7) is applied and f contributes $\frac{1}{7}$. Hence s_{-1} is hungry. Therefore, s_{-1} is a 3-face and s_0 is an isolated 3-face. If s_{-1} is a bad triangle, then s_{-1}, s_0, t form (C13) which is reducible. Hence s_{-1} is not bad and it forms a diamond with a bad 3-face t_{-1}. See Figure 6(b). If both vertices shared by s_{-1} and t_{-1} have degree four, faces s_{-1}, t_{-1}, s_0, t form configuration (C15). If both shared vertices have degree at least 5, then the diamond has nonnegative charge after the vertex rules so s_{-1} cannot be hungry.

Hence one shared vertex w has degree 5 and the other is of degree four. If w is not incident to any other bad 3-face, then the faces s_{-1} and t_{-1} are not hungry. If w is in a bad 3-face t', then t', s_{-1}, t_{-1}, s_0, and t form a copy of (C16). Therefore, (R7) applies and contributes $\frac{1}{7}$.

Thus $\mu^*(t) \geq -3 + 1 + 1 + 3 \cdot \frac{1}{7} + 3 \cdot \frac{1}{7} + \frac{1}{7} \geq 0$. □

4.3 Reducibility

We now show that any minimum counterexample (G, L) to Theorem 6 is prime. Since we already proved that no pair (G, L) is prime, this shows that no counterexample exists.

We start by proving basic properties of G. If G is not connected, there exists an L-coloring for every connected component of G which together give an L-coloring of G. Hence G is connected.

Suppose that G has a cut-vertex v. Let G_1 and G_2 be proper subgraphs of G such that $G_1 \cap G_2 = v$, $G = G_1 \cup G_2$ and $p \in G_1$. By the minimality of G, there exists an L-coloring φ of G_1. Let L' be lists on G_2 where $L'(v) = \varphi(v)$ and $L'(u) = L(u)$ for $u \in V(G_2) - v$. By the minimality of G, there exists an L'-coloring ψ of G_2. Colorings φ and ψ together give an L-coloring of G, a contradiction. Hence G is 2-connected.

Suppose $v \in V(G) - p$ has degree at most two. An L-coloring φ of $G - v$ can be extended to v since $|L(v)| \geq 3$. Hence $d(v) \geq 3$ for every $v \in V(G) - p$.

Suppose $d(p) = 1$. Let v be the neighbor of p. Since G is 2-connected, v is not a cut-vertex. So $V(G) = \{p, v\}$ and G is L-colorable. Hence $d(p) \geq 2$.

By the minimality of G, lists of endpoints of every edge $e \in E(G)$ have a color in common. If not, e can be removed from G without changing possible L-colorings. We denote the color shared by the endpoints of e by $c(e)$.

Lemma 9. *There is no vertex v with a color $c \in L(v)$ not appearing on the edges incident to v.*

Proof. Suppose $v \in V(G)$ has a color $c \in L(v)$ not appearing in the lists of the adjacent vertices. Let φ be an L-coloring of $G - v$. An L-coloring of G can be obtained by assigning assigning $\varphi(v) = c$. □

Lemma 10. *G does not contain a trail of three edges $e_1e_2e_3$ where $c(e_1) = c(e_3) \neq c(e_2)$.*

Proof. Suppose G contains a trail of three edges $e_1e_2e_3$ where $c(e_1) = c(e_3) \neq c(e_2)$. The lists of the two endpoints of e_2 both contain the colors $c(e_1)$ and $c(e_2)$, which is a contradiction to the L-list assignment if $c(e_1) \neq c(e_2)$. □

Lemma 11. *G does not contain a 3-face $e_1e_2e_3$ incident to a low vertex with $c(e_1) = c(e_2)$.*

Proof. If v is a low vertex in a 3-face bounded by $e_1e_2e_3$, then by Lemma 9 the edges incident to v have distinct colors. Thus, if $c(e_1) = c(e_2)$, they are not both incident to v and $c(e_3) \neq c(e_1) = c(e_2)$, and thus a trail from Lemma 10 is contained in G. □

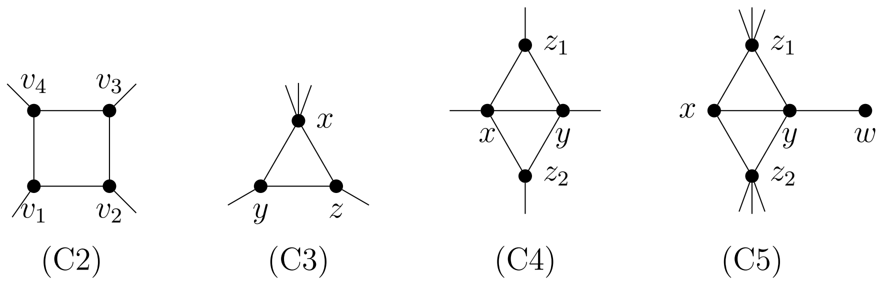

Figure 7: Situations in Lemmas 14, 15, 16, and 17.

Lemma 12. *G contains no copy of (C1).*

Proof. Suppose puv is a 3-face. Let $c = L(p) = c(pu) = c(pv)$. By Lemma 10, also $c(uv) = c$. Let φ be and L-coloring of $G - uv$. Since $\varphi(p) = c$, $\varphi(u) \neq c$ and $\varphi(v) \neq c$. Hence $\varphi(u) \neq \varphi(v)$ and φ is an L-coloring of G. $\square$

We will show that a minimum counterexample (G, L) contains no copy of the configurations (C2)–(C16) by using a concrete form of reducibility.

Definition 13. A configuration C is *reducible* if there exist disjoint sets $X, R \subseteq V(C)$ where X is nonempty, $p \notin X \cup R$, the set $X \cup R$ contains exactly the vertices of C with at most one neighbor outside C, and for every L-coloring φ of $G - X$, there exists an L-coloring ψ of G satisfying the following properties:

- $\varphi(v) = \psi(v)$ for all $v \notin X \cup R$,
- if $\varphi(x) \neq \psi(x)$ for some $x \in R$ with one neighbor outside of C, then $|L(x) \cap \{\psi(y) : y \in N(x) \cap V(C)\}| \leq 1$, and
- if $\varphi(x) \neq \psi(x)$ for some $x \in R$ with no neighbor outside of C, then $|L(x) \cap \{\psi(y) : y \in N(x) \cap V(C)\}| \leq 2$.

If a graph G contains a copy of a reducible configuration, then it is not a minimum counterexample since L-colorings of proper subgraphs extend to L-colorings of G. We now use this definition to prove our simple configurations (C2)–(C6) are reducible.

Lemma 14. *(C2) is reducible.*

Proof. Let $F = v_1v_2v_3v_4$ be a normal 4-face where $d(v_i) = 3$ for $i \in \{1, 2, 3, 4\}$; see Figure 7. Let $X = F$ and $R = \varnothing$ and let φ be an L-coloring of $G - X$. Each vertex v_i has at most one color forbidden by φ, which implies that $|L_\varphi(v_i)| \geq 2$. The only case when a cycle is not 2-choosable is when the cycle has odd length and each vertex has the same list of size 2. Hence φ can be extended to an L-coloring ψ of G. $\square$

Lemma 15. *(C3) is reducible.*

Proof. Let xyz be a 3-face where y and z have degree 3; see Figure 7. Let $X = \{y, z\}$ and $R = \varnothing$, and let φ be an L-coloring of $G - X$. By Lemmas 9 or 10, the color $\varphi(x)$ is not equal to both $c(xy)$ and $c(xz)$. Thus, without loss of generality, we assume $\varphi(x) \neq c(xz)$. Observe $|L_\varphi(y)| \geq 1$ and $|L_\varphi(z)| \geq 2$, and thus we can color y and then z to find an L-coloring ψ of G. □

Lemma 16. *(C4) is reducible.*

Proof. Let z_1xyz_2 be a diamond as in (C4); see Figure 7. Let $X = \{z_1, x, y, z_2\}$ and $R = \varnothing$, and let φ be an L-coloring of $G - X$. Observe $|L_\varphi(v)| \geq 2$ for all $v \in \{z_1, x, y, z_2\}$. If the color $c(xy)$ no longer appears in both $L_\varphi(x)$ and $L_\varphi(y)$, then we can remove the edge xy and extend the coloring to an L-coloring ψ of G since z_1xz_2y is a 4-cycle, which is 2-choosable.

Suppose that $c(xy) \in L_\varphi(x)$. By Lemma 11, $c(xy) \neq c(xz_1)$ and $c(xy) \neq c(xz_2)$. Set $\varphi(x) = c(xy)$. Now φ can be extended to an L-coloring of G by coloring y and then z_1, z_2 in a greedy way.

□

Lemma 17. *(C5) is reducible.*

Proof. Let z_1xyz_2 be a diamond as in (C5); see Figure 7. Let $X = \{x\}$ and $R = \{y\}$, and let φ be an L-coloring of $G - X$. By Lemma 9, the colors $c(xv)$ are distinct for $v \in \{z_1, y, z_2\}$. If $\varphi(v) \neq c(xv)$ for some $v \in \{z_1, y, z_2\}$, then we color $\varphi(x) = c(xv)$ to find an L-coloring of G without recoloring $y \in R$. Thus, $\varphi(v) = c(xv)$ for all $v \in \{z_1, y, z_2\}$.

By Lemma 10, the color $c(z_ix)$ is distinct from $c(z_iy)$ for each $i \in \{1, 2\}$. Thus, the colors $\varphi(z_i)$ are not in $L(y)$, so $|L(y) \cap \{\varphi(v) : v \in \{x, z_1, z_2\}\}| = 1$. Thus there is at least one color $a \in L(y)$ other than $c(xy)$ and $c(yw)$, where w is the neighbor of y outside the diamond. We color $\psi(x) = c(xy)$ and recolor $\psi(y) = a$ to find an L-coloring of G. □

Lemma 18. *(C6) is reducible.*

Proof. Observe that (C6) contains (C5) as a subgraph and the proof for (C5) works also for (C6). □

To complete the list of reducible configurations, we describe a way to build compound reducible configurations from simple reducible configurations by adding a bad face.

Lemma 19 (Iterative Construction)**.** *Let C be a reducible configuration and let $v \in V(C)$ have a unique neighbor $u \in N(v) \setminus V(C)$. Let C' be obtained from C by removing the edge vu and adding two new vertices x, y such that vxy is a 3-face, y has exactly one neighbor z in $N(y) \setminus V(C')$ and x has at least one neighbor in $N(x) \setminus V(C')$. If C is reducible, then C' is reducible.*

See Figure 8 for a visualization of this construction.

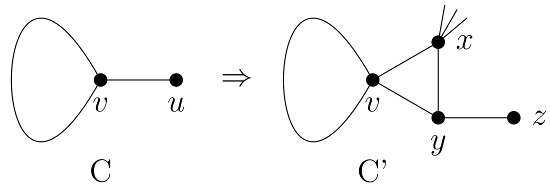

Figure 8: Creating compound reducible configurations in Lemma 19.

Proof. Let G contain C'. Observe that since y has degree three, Lemmas 9 and 10 guarantee that $c(vy)$, $c(yx)$, and $c(vx)$ are distinct.

Let $X, R \subseteq V(C)$ be given by the definition of C being reducible. Let $X' = X$ and $R' = R \cup \{y\}$. We consider cases based on whether v is in X or R.

Case 1: $v \in X$ For an L-coloring φ of $G - X$, we use the method of extending a coloring to the vertices in C given by its proof of reducibility. When coloring v, the method expects only one color from $L(v)$ appearing in its neighbors outside of C. If at most one of $\varphi(x)$ or $\varphi(y)$ appears in $L(v)$, then the method to color C completes with an L-coloring ψ of G. Otherwise, $\varphi(y) = c(vy)$ and $\varphi(x) = c(vx) \neq c(yx)$. Thus, we recolor $\psi(y) = c(yx)$ and assign $\psi(v) = \varphi(y)$. This recolors y with a color that does not appear in its neighbors, and y has exactly one color restricted within C'.

Case 2: $v \in R$ For an L-coloring φ of $G - X'$, we use the method of extending a coloring to the vertices in C given by its proof of reducibility. If it can be colored without recoloring v, then the resulting coloring is an L-coloring on G. However, if v must be recolored, then v has at most one color restricted from within C. Since $c(vx) \neq c(vy)$, if v has no available colors for this recoloring, we have $\varphi(y) = c(vy)$ and $\varphi(x) = c(vx) \neq c(xy)$. Thus, we recolor $\psi(v) = \varphi(y) = c(vy)$ and $\psi(y) = c(yx)$. Observe that v has at most two colors restricted by its neighbors in C', and y has at most one color restricted by its neighbors in C'.

In either case, we have modified the coloring algorithm for C to apply for C'. □

Lemma 20. *(C7)–(C16) are reducible.*

Proof. Each configuration is built using Lemma 19 from a known reducible configuration. We use the notation "(Ci) $\longrightarrow$ (Cj)" to denote "Applying Lemma 19 to (Ci) results in (Cj)," in the following pairs:

(C6)	$\longrightarrow$	(C7)	(C4)	$\longrightarrow$	(C8)	(C5)	$\longrightarrow$	(C9)
(C4)	$\longrightarrow$	(C10)	(C2)	$\longrightarrow$	(C11)	(C3)	$\longrightarrow$	(C12)
(C12)	$\longrightarrow$	(C13)	(C8)	$\longrightarrow$	(C14)	(C8)	$\longrightarrow$	(C15)
			(C15)	$\longrightarrow$	(C16)			

Thus, by Lemma 19 and previous lemmas, these configurations are not in G. □

5 Conclusion

The main problem if planar graphs are $(3, 1)$-choosable remains open. We hope that this paper could serve as an inspiration of possible approaches to the problem. Unfortunately, the conditions of Theorem 4 and Theorem 5 are not valid for all planar graphs; see Figure 9. Let us note that we do not have an example where P has two vertices that are not in a triangle.

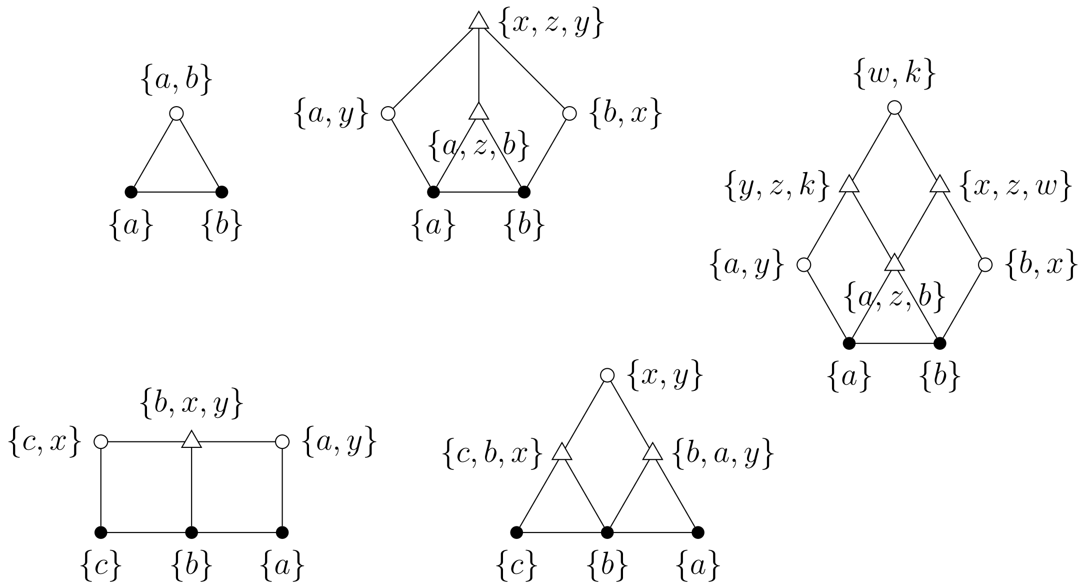

Figure 9: Some examples where conditions of Theorem 4 and Theorem 5 do not generalize to all planar graphs.

The last thing we promised is a planar graph G without 4-cycles and 5-cycles that is not $(3, 2)$-choosable. It is a modification of construction of Wang, Wen, and Wang [11]. The main building gadget is the graph H depicted in Figure 10. It has two vertices with lists of size one. The graph G is created by taking 9 copies of H and identifying vertices with lists $\{a\}$ into one vertex v and vertices with lists $\{b\}$ into one. Vertices u and v get disjoint lists and we assign to every of 9 possible coloring of u and v one gadget, where the coloring of u and v cannot be extended. By inspecting the gadget, reader can check that G cannot be colored and that G has no cycles of length 4 or 5.

The authors thank Mohit Kumbhat for introducing them to the problem during 3rd Emléktábla Workhops and thank Kyle F. Jao for fruitful discussions and encouragement in the early stage of the project.

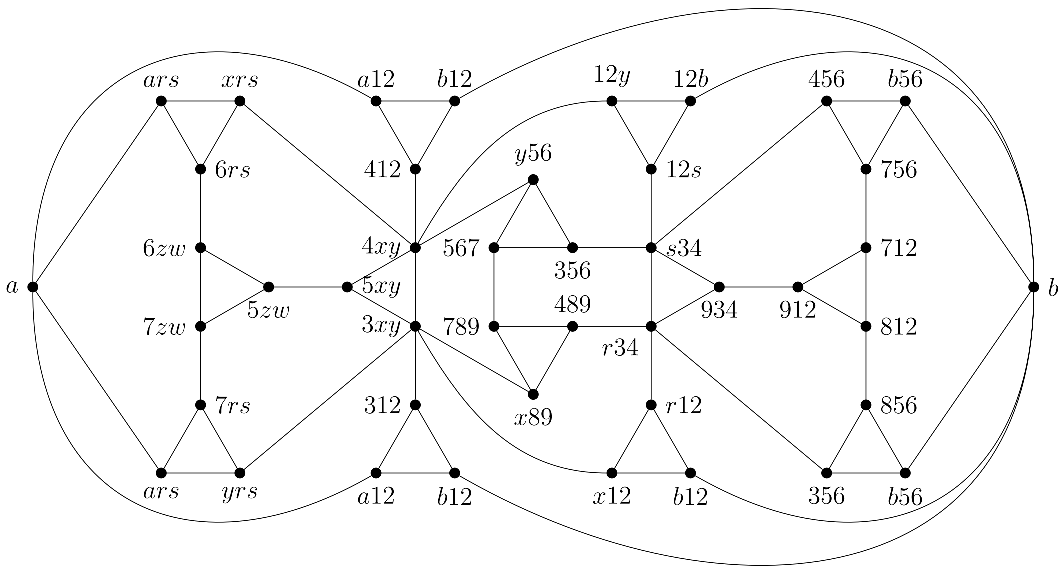

Figure 10: Building block of a non $(3, 2)$-choosable planar graph without cycles of length 4 and 5.

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