Shortened Universal Cycles for Permutations

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Abstract

Kitaev, Potapov, and Vajnovszki [On shortening u-cycles and u-words for permutations, Discrete Appl. Math, 2019] described how to shorten universal words for permutations, to length $n!+n-1-i(n-1)$ for any $i \in [(n-2)!]$, by introducing incomparable elements. They conjectured that it is also possible to use incomparable elements to shorten universal cycles for permutations to length $n! - i(n-1)$ for any $i \in [(n-2)!]$. In this note we prove their conjecture. The proof is constructive, and, on the way, we also show a new method for constructing universal cycles for permutations.

1 Introduction

A *universal cycle* for a family $\mathcal{F}$ of combinatorial objects is a cyclic sequence whose consecutive substrings of a given length n represent each object of $\mathcal{F}$ exactly once. The canonical type of universal cycle is a De Bruijn cycle, which is a universal cycle for the words of length n over an alphabet $\mathcal{A}$. De Bruijn cycles have been widely studied [6]. Chung, Diaconis, and Graham [4] introduced the notion of universal cycles for other combinatorial objects such as permutations, sets, and set partitions. A *universal word* for $\mathcal{F}$ is the non-cyclic analogue of a universal cycle. A universal cycle has length $|\mathcal{F}|$, while a universal word has length $|\mathcal{F}| + (n-1)$.

In this paper, we are focusing on permutations. We use $[n]$ to denote $\{1, 2, \ldots, n\}$ and S_n to denote all permutations of $[n]$. A *universal word* for S_n is a word w over $\mathbb{N}$ such that each permutation in S_n is order-isomorphic to exactly one consecutive subword of length n. Notice that entries in w are not restricted to $[n]$. For example, 14524314 is a universal word for S_3. The permutations in S_3 represented by w from left to right are 123, 231, 312, 132, 321, and 213. A *universal cycle* for S_n is a cyclic universal word. For example, 145243 is a universal cycle for S_3. Note that a universal word has length $n!+(n-1)$, and a universal cycle has length $n!$. Hurlbert [11] showed that universal cycles for S_n exist for all n. Chung, Diaconis, and Graham [4] conjectured that it is sufficient to use $n+1$ distinct numbers in a universal cycle for S_n, which would be best possible for $n \geq 3$. This conjecture was proved by Johnson [15]. Universal cycles for permutations have applications in various areas such as molecular biology [5], computer vision [14], robotics [16], and psychology [17].

Universal cycles, for permutations and more generally for $\mathcal{F}$, are useful because they represent the elements of $\mathcal{F}$ compactly. In recent years there has been interest in shortening universal cycles to compress information even further. (These efforts are not to be confused with the study of shorthand universal cycles for permutations, which have the same lengths as universal cycles for S_n but only n distinct symbols;

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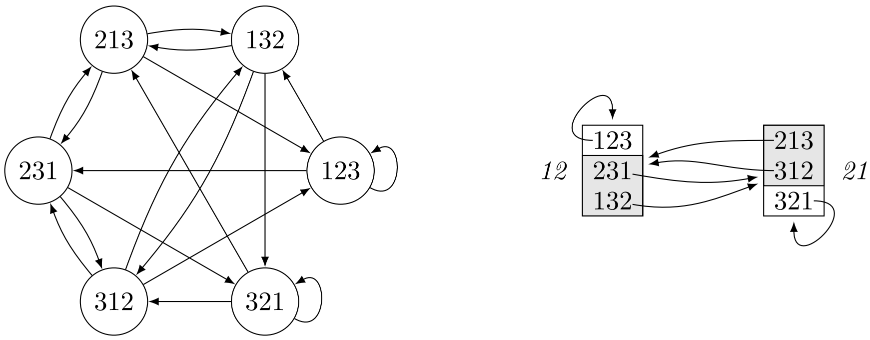


Figure 1: The graph of overlapping 3-permutations on the left and the cluster graph for 3-permutations on the right.

see [10].) De Bruijn cycles have been shortened, to *universal partial cycles* and *universal partial words*, using a wildcard symbol $\diamond$ that covers any letter of the alphabet, so that a window of length n may cover more than one word of length n; see [3,8,9]. Graph universal cycles, introduced in [1], and graph universal cycles for permutations, introduced in [2], have been shortened in [12].

Kitaev, Potapov, and Vajnovszki [13] shortened universal words for permutations in two different ways: with wildcard symbols and with incomparable elements. Similarly to universal partial words and universal partial cycles, they considered using the wildcard symbol $\diamond$, which yielded many nonexistence results, but they created shortened universal words for permutations using a wildcard symbol $\diamond_{\{a,b\}}$ that covers either of the two elements a and b.

Most relevantly to our work, in [13], Kitaev, Potapov, and Vajnovszki used incomparable elements at distance $n-1$ to shorten universal words for permutations of $[n]$ to lengths $n! + n - 1 - i(n-1)$ for each $i \in [(n-2)!]$. A word w of length n with incomparable elements *covers* all permutations of length n that are linear extensions of the order given by w. In the case of incomparable elements at distance $n-1$, w covers two permutations. For example, the word 2132 covers the two permutations 2143 and 3142. We also say that a longer word v containing w as a consecutive substring *covers* the permutations that w covers. For example, 4321324 contains 2132 as a consecutive substring, so 4321324 covers 2143 and 3142; the other 4-permutations that 4321324 covers are 4321, 3214, 4213, and 1324.

Theorem 1 (Kitaev, Potapov, and Vajnovszki [13])**.** *Using incomparable elements at distance $n-1$, one can obtain shortened universal words for S_n of lengths $n! + n - 1 - i(n-1)$ for each $0 \le i \le (n-2)!$.*

They conjecture [13, Conjecture 8] that their result may be strengthened by obtaining shortened universal cycles instead of shortened universal words. Here we prove their conjecture.

Theorem 2. *For $n \ge 3$ and each $0 \le i \le (n-2)!$, using incomparable elements at distance $n-1$, one can obtain a shortened universal cycle for S_n of length $n! - i(n-1)$.*

Getting from Theorem 1 to Theorem 2 requires several background definitions which we provide forthwith. For a word w over comparable letters, we denote by $\mathrm{red}(w)$ the word obtained from w by replacing each copy of the ith smallest element of w by i.

For any n, the *graph of overlapping n-permutations* is a directed graph on $n!$ vertices, each vertex corresponding to one permutation in S_n. There is an edge from x to y iff $\mathrm{red}(x_2x_3\cdots x_n) = \mathrm{red}(y_1y_2\cdots y_{n-1})$, see Figure 1. Notice that a universal cycle for S_n gives a Hamiltonian cycle in the graph of overlapping n-permutations.

The cluster graph, where a universal cycle corresponds to an Eulerian tour rather than a Hamiltonian cycle, has also been used to study universal cycles for S_n; see [4]. The *cluster graph* for n-permutations, denoted here by G, identifies all permutations x and y where $\mathrm{red}(x_1\cdots x_{n-1}) = \mathrm{red}(y_1\cdots y_{n-1})$. This means the edges are grouped by their origin. Its vertices are the clusters of n-permutations whose first $n-1$ entries are order-isomorphic, and its edges are the n-permutations: it has an edge from σ to τ for each n-permutation x satisfying $\sigma = \mathrm{red}(x_1 \cdots x_{n-1})$ and $\tau = \mathrm{red}(x_2 \cdots x_n)$. This means that each cluster contains n permutations, and the cluster graph is a directed multigraph. Compressing the parallel edges of the cluster graph for n-permutations to single edges yields the graph of overlapping $(n-1)$-permutations. For example, the cluster graph for 4-permutations is shown in Figure 2, and after compressing parallel edges it is the same as the graph of overlapping 3-permutations shown on the left of Figure 1. The cluster graph G is balanced and strongly connected, as observed in [4].

Any Eulerian tour in the cluster graph gives a Hamiltonian cycle in the graph of overlapping n-permutations. It is conjectured that these Hamiltonian cycles can be extended to universal cycles for n-permutations; see [4, 11].

For De Bruijn sequences, translating a Hamiltonian cycle in the De Bruijn graph to a De Bruijn cycle is straightforward since the universal cycle uses just n letters. This is not the case for n-permutations. A universal cycle or a universal word for n permutations may use many more distinct entries than n. Building a universal cycle for n-permutations by following a Hamiltonian cycle in the graph of overlapping n-permutations could possibly lead to a situation where the beginning and the end are not compatible. To illustrate this potential misalignment, consider the following simple example. Suppose $n = 4$ and we start with $1, 2, 3, 4$. When we end walking along the cycle, we have a universal word w that begins with 1234 and ends with 678. Now it is not possible to turn w directly into a universal cycle since the beginning 123 and the end 678 of w are different. It is, however, easy to construct a universal word by following a Hamiltonian path.

The proof of Theorem 1 utilizes the cluster graph of n-permutations and performs a compression on the cluster graph. We use the same shortening ideas as Kitaev, Potapov, and Vajnovszki [13]. In order to obtain a universal cycle instead of a universal word, we identify a particular part of the cluster graph that allows for the beginning and the end of the word coming from traversing an Eulerian tour to be connected. We use many of the lemmas as Kitaev, Potapov, and Vajnovszki [13], we include also proofs for completeness.

2 Proof of Theorem 2

We start by introducing and restating some technical definitions. Then we describe several properties of the cluster graph from Kitaev, Potapov, and Vajnovszki [13]. Finally, we describe the procedure allowing us to obtain a universal cycle.

In the cluster graph G for n-permutations, for an edge e, we write $L(e)$ for the corresponding permutation. For added flexibility later, we consider $L : E(G) \to \mathbb{N}^n$ to assign words. Later in this section, we will replace some permutations by words in the labels of the cluster graph. This is used for shortening.

The cluster graph G is balanced and strongly connected, as observed in [4].

Two permutations $\pi_1 \cdots \pi_n$ and $\sigma_1 \cdots \sigma_n$ are called *twins* if they belong to the same cluster and $|\pi_1 - \pi_n| = |\sigma_1 - \sigma_n| = 1$. Examples of twins are 3142 and 2143, 134562 and 234561. Pairs of twins are depicted in Figure 2 in gray boxes, and they correspond to a pair of parallel edges.

The proof of Theorem 2 is constructive. Here is a sketch of the proof, omitting compression. In the cluster graph G, we identify a cycle given by permutations $\mathcal{P}$ defined in (1). We remove them from G and find a word w representing $G - \mathcal{P}$. In order to do that, we need that $G - \mathcal{P}$ is Eulerian. Lemma 5 guarantees $G - \mathcal{P}$ is connected, and Lemma 6 shows how to build w. Finally, we extend w to a cyclic word that includes $\mathcal{P}$ in Lemma 7.

A series of Lemmas in [13] describe additional properties of the cluster graph. We include their proofs for completeness.

Lemma 3 (Kitaev, Potapov, and Vajnovszki [13]). *The following are true for each cluster graph G.*

(i) Each cluster has exactly one pair of twins.

(ii) For each cluster X, there exists a unique cluster Y such that there are two edges from X to Y. Also, there are no clusters X and Y such that there are three or more edges from X to Y.

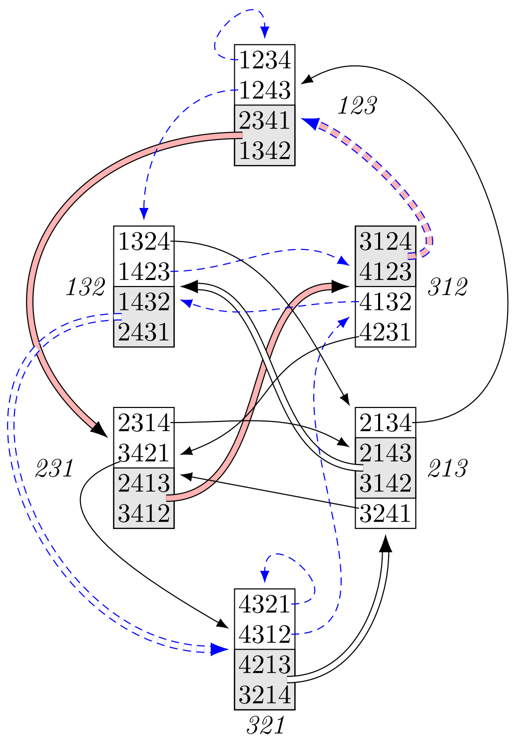


Figure 2: Cluster graph for 4-permutations. Grey boxes indicate pairs of twins. A double edge represents parallel edges corresponding to the permutations in the gray box. Double edges with filled insides and white insides form two cycles that could be used for shortening the universal cycle. Blue dashed edges denote permutations in $\mathcal{P}^{\star}$. They are used in the gluing part depicted in Figure 6.

(iii) For each cluster Y, there exists a unique cluster X such that there are two edges from X to Y.

(iv) Any of the disjoint cycles formed by the double edges goes through exactly $n-1$ distinct clusters.

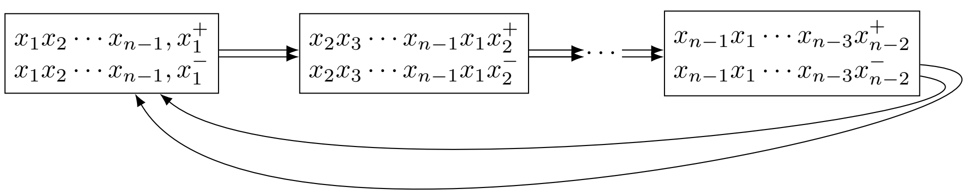
Proof. For an integer x, we denote by x^+ and x^- real numbers such that $x < x^+ < x+1$ and $x-1 < x^- < x$.

(i) Let $x_1x_2\dots x_{n-1}$ be a vertex in G. The only two parallel edges leaving this vertex correspond to $\mathrm{red}(x_1x_2\dots x_{n-1}x_1^+)$ and $\mathrm{red}(x_1x_2\dots x_{n-1}x_1^-)$.

(ii) Two permutations in the same cluster can only have the last $(n-1)$ elements isomorphic when two permutations are twins. By (i), there are exactly two twins in each cluster, and they form parallel edges to another cluster.

(iii) Let $x_1x_2\dots x_{n-1}$ be the permutation Y. Then, the twins $\mathrm{red}(x_{n-1}^-x_1x_2\dots x_{n-1})$ and $\mathrm{red}(x_{n-1}^+x_1x_2\dots x_{n-1})$ correspond to parallel edges to Y.

(iv) Since double edges are formed exactly by twin permutations, the only cycles formed by double edges have the following form:



□

Let $\mathcal{P}$ be the following set of permutations:

$$\mathcal{P} = \left\{ \begin{array}{ll} (n, n-1, \dots, 2, 1), & (1, 2, \dots, n-2, n, n-1), \\ (n, n-1, \dots, 3, 1, 2), & (1, 2, \dots, n-3, n, n-2, n-1), \\ (n, n-1, \dots, 4, 1, 2, 3), & (1, 2, \dots, n-4, n, n-3, n-1, n-2), \\ \vdots & \vdots \\ (n, n-1, \dots, k, 1, \dots, k-1), & (1, 2, \dots, k, n, k+1, n-1, n-2 \dots, k+2), \\ \vdots & \vdots \\ (n, n-1, 1, 2, \dots, n-1), & (1, n, 2, n-1, n-2, \dots, 3), \\ (n, 1, 2, \dots, n-1), & (n, 1, n-1, n-2, \dots, 2), \\ (1, 2, \dots, n), & (1, n, n-1, n-2, \dots, 2) \end{array} \right\}. \tag{1}$$

Observe that permutations in $\mathcal{P}$ when read first column top to bottom followed by the second column top to bottom form a tour in the cluster graph. We will use $\mathcal{P}$ for gluing the ends of a universal word for $S_n \setminus \mathcal{P}$ to create a universal cycle for S_n. To accommodate shortening, we also need to consider twins of permutations in $\mathcal{P}$. The only permutations in $\mathcal{P}$ that have twins are $(n, 1, 2, \dots, n-1)$ and $(1, n, n-1, n-2, \dots, 2)$. Let

$$\mathcal{P}^\star = \mathcal{P} \cup \{(n-1, 1, 2, 3, \dots, n-2, n), (2, n, n-1, \dots, 4, 3, 1)\}.$$

Hence $\mathcal{P}^\star$ is obtained from $\mathcal{P}$ by adding twin permutations to those already in $\mathcal{P}$. Blue dashed edges in Figure 2 correspond to $\mathcal{P}^\star$.

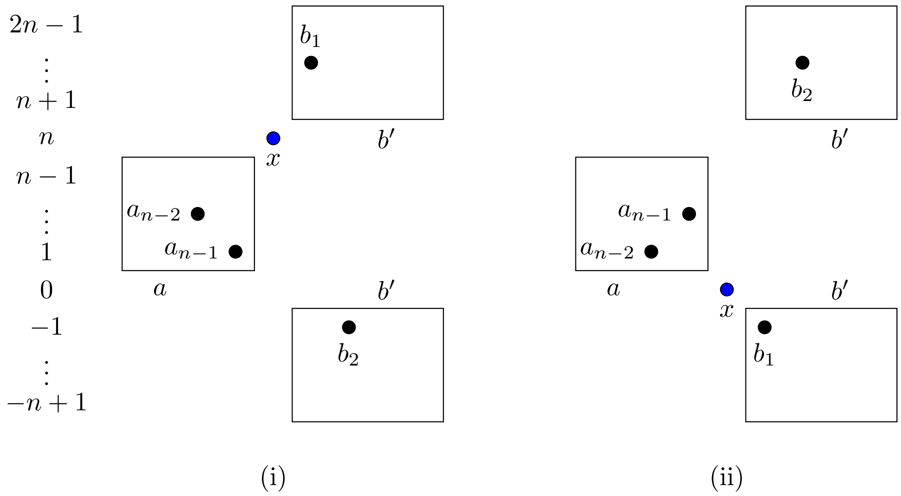


Figure 3: Transitioning between two clusters in Lemma 5. (i) depicts $a_{n-1} < a_{n-2}$ and $b_1 > b_2$ and (ii) depicts $a_{n-1} > a_{n-2}$ and $b_1 < b_2$. The remaining two cases are similar.

Claim 4. *For $n \geq 4$, the permutations in $\mathcal{P}^\star$ do not contain 213 or 231 as a consecutive subpermutation.*

Proof. Inspection of cases. $\square$

If TW is a set of twins in a cluster graph G, the *compressed cluster graph* for TW is obtained from G by replacing each pair of parallel edges in TW, labeled by L as $(x_1x_2\cdots x_{n-1}x_n)$ and $(x_nx_2\cdots x_{n-1}x_1)$, by one edge e with $L(e) = \text{red}(x_1x_2\cdots x_{n-1}x_1)$. See Figure 5 for an illustration where TW forms a cycle.

Lemma 5. *For $n \geq 4$, the graph obtained from a compressed cluster graph G by removing all edges corresponding to $\mathcal{P}^\star$ is strongly connected.*

Proof. To show $G - \mathcal{P}^\star$ is strongly connected, we will prove there exists a walk between any two vertices in $G - \mathcal{P}^\star$. Notice that a compressed graph is obtained by replacing parallel edges by single edges. Hence for simplicity, we assume G is not compressed.

Let $a = a_1a_2\cdots a_{n-1}$ and $b = b_1b_2\cdots b_{n-1}$ be permutations in S_{n-1} corresponding to two vertices in $G - \mathcal{P}^\star$. We construct a word P that corresponds to a walk in $G - \mathcal{P}^\star$ starting at a and ending at b. We define $P = axb'$, where x is an integer and b' is created from b as follows:

$$x = \begin{cases} n & \text{if } a_{n-2} > a_{n-1}, \\ 0 & \text{if } a_{n-2} < a_{n-1}, \end{cases} \qquad b'_i = \begin{cases} b_i + n & \text{if } b_i > (b_1+b_2)/2, \\ b_i - n & \text{otherwise.} \end{cases}$$

See Figure 3 for sketch of the construction of a general P and for a particular example see Figure 4. Notice that $\text{red}(b') = b$ and $\{\text{red}(a_{n-2}a_{n-1}x), \text{red}(xb_1b_2)\} \subseteq \{231, 213\}$. Hence any n-permutation in P contains x and contains a consecutive triple that reduces to 213 or 231. Neither of these are contained in any permutation in $\mathcal{P}^\star$ by Claim 4. Therefore, P corresponds to a path in $G - \mathcal{P}^\star$ from a to b, and hence $G - \mathcal{P}^\star$ is strongly connected. $\square$

For a word w, an *n-window* of w is a subword of consecutive letters of length n.

Lemma 6. *Let $T = (e_1, e_2, \ldots, e_\ell)$ be a trail in a compressed cluster graph for n-permutations. There exists a word $w = w_1\cdots w_{\ell+n-1}$, where $\text{red}(w_i\cdots w_{i+n-1}) = L(e_i)$ for all $1 \leq i \leq \ell$.*

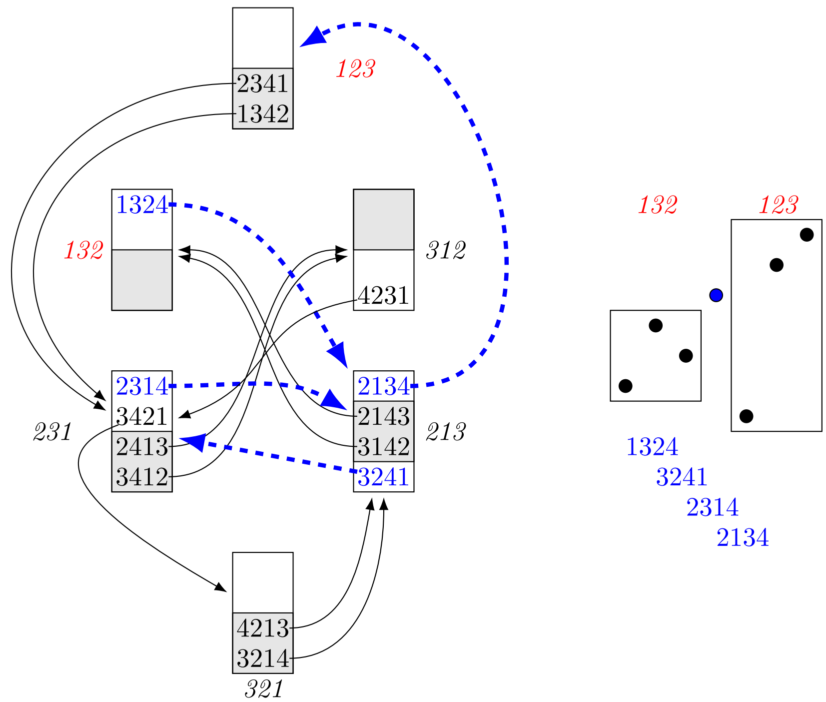


Figure 4: Cluster graph for 4-permutations with $\mathcal{P}^\star$ deleted. This figure demonstrates using the procedure in Lemma 5 to obtain a trail from *132* to *123*, depicted using dashed edges.

Proof. If $\ell = 1$, we let $w = L(e_1)$. If $\ell \geq 2$, let $w' = w'_1 \cdots w'_{\ell+n-2}$ be a word for $(e_1, e_2, \ldots, e_{\ell-1})$ by induction. Let $a_1 a_2 \cdots a_n$ be $L(e_\ell)$. If $a_n = a_1$, i.e. e_ℓ is a compressed edge, then w is obtained from w' by appending w'_ℓ, which is the letter corresponding to a_1. Otherwise, $a_1 \neq a_n$, and we may need to modify w' and determine a letter to append as follows.

If $a_n = n$, let $x = \max\{w'_i : 1 \leq i \leq \ell+n-2\} + 1$. If $a_n < n$, let i be the index such that $a_i = a_n + 1$ and $x = w'_{\ell-1+i}$ be the letter in w' corresponding to a_i. Then we define $w = w_1 \cdots w_{\ell+n-1}$ as

$$w_i = \begin{cases} w'_i & \text{if } w'_i < x \text{ and } i \leq \ell+n-2, \\ w'_i + 1 & \text{if } w'_i \geq x \text{ and } i \leq \ell+n-2, \\ x & \text{if } i = \ell+n-1. \end{cases}$$

Since all n-windows of $w_1 \cdots w_{\ell+n-2}$ are order isomorphic to n-windows of $w'_1 \cdots w'_{\ell+n-2}$, w still represents the same permutations as w' in the part before the last letter. The choice of x makes the last n-window in w be order isomorphic to $L(e_\ell)$. □

The next lemma describes how to use $\mathcal{P}$ to turn a particular word w into a cyclic word z. It adds $n+1$ letters at the beginning of w. The result is a cyclic word covering the permutations in w and also all permutations in $\mathcal{P}$.

Lemma 7. *Let $n \geq 4$. Let $\mathcal{P}'$ be such that $\mathcal{P} \subseteq \mathcal{P}' \subseteq \mathcal{P}^\star$. Let $w = w_1 w_2 \cdots w_k$ be a word with $k \geq n-1$ that covers each n-permutation in S_n at most once, and let $\mathcal{W}$ be the set of n-permutations covered by w. If $red(w_1 w_2 \cdots w_{n-1}) = red(w_{k-n+2} \cdots w_{k-1} w_k) = (n-1, \ldots, 2, 1)$ and $\mathcal{W} \cap \mathcal{P}' = \emptyset$, then there exists a cyclic word z of length $k+n+1$ such that each permutation in $\mathcal{W} \cup \mathcal{P}'$ is covered by exactly one n-window of z. The permutations in $\mathcal{P}' \setminus \mathcal{P}$ are covered in a compressed way.*

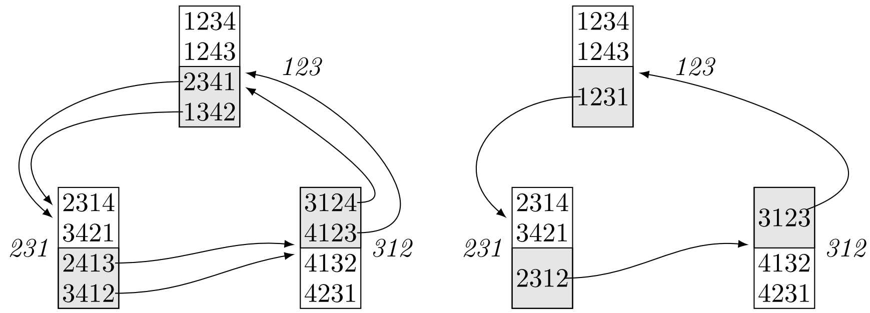


Figure 5: The effect of compressing one cycle in the cluster graph for $n = 4$. See Figure 2 for the entire cluster graph.

Proof. The cyclic word $z = z_1 z_2 \cdots z_{n+1+k}$ can be defined as follows; see Figure 6 for guidance:

$$\begin{aligned}
z_i &= \min\{w_1, w_2, \ldots, w_k\} - n - 1 + i \text{ for } 1 \le i \le n-1, \\
z_n &= \max\{w_1, w_2, \ldots, w_k\} + 1, \\
z_{n+1} &= \begin{cases} z_{n-1} + 1 & \text{if } (2, n, n-1, \ldots, 4, 3, 1) \notin \mathcal{P}', \\ w_{n-1} & \text{otherwise,} \end{cases} \\
z_{n+1+i} &= w_i \text{ for } 1 \le i \le k-1, \\
z_{n+1+k} &= \begin{cases} w_k & \text{if } (n-1, 1, 2, 3, \ldots, n-2, n) \notin \mathcal{P}', \\ z_{n-1} & \text{otherwise.} \end{cases}
\end{aligned}$$

The n-windows of the cyclic word z that start at indices $k+3$ through $n+1$ (the first $n-1$ of which wrap around to the beginning of z) cover each permutation in $\mathcal{P}'$ exactly once, in the order given by reading the left column in (1) first and the right column second, and with the permutations in $\mathcal{P}' \setminus \mathcal{P}$ covered in a compressed way. Notice that the possible shortenings are obtained by n-windows starting at z_{n+1} and z_{n+1+k}.

Since $z_{n+2} \cdots z_{n+k} = w_1 \cdots w_{k-1}$, all permutations in $\mathcal{W}$ except the last one containing w_k are covered by z. In the case that $z_{n+1+k} = w_k$, the last one is there as well. We need to check that $\mathrm{red}(w_{k-n+1} \cdots w_k) = \mathrm{red}(z_{k+2} \cdots z_{n+1+k})$ when $z_{n+1+k} = z_{n-1}$. By hypothesis, $w_{k-n+2} > \cdots > w_{k-1} > w_k$ so, if $w_k \ge w_{k-n+1}$, then $\mathrm{red}(w_{k-n+1} \cdots w_k) = (1, n, n-1, \ldots, 2) \in \mathcal{P}$ or $\mathrm{red}(w_{k-n+1} \cdots w_k) = (1, n-1, n-2, \ldots, 1)$, in both cases contradicting $\mathcal{P} \cap \mathcal{W} = \emptyset$. Therefore, $w_k < w_{k-n+1}$, so $w_k < \min\{w_{k-n+1}, \ldots, w_{k-1}\}$, and decreasing w_k to $z_{n+1+k} = z_{n-1} = \min\{w_1, w_2, \ldots, w_k\} - 2$ does not change the relative order of $w_{k-n+1} \cdots w_k$. □

Now we are ready to prove Theorem 2.

Theorem 2. *For $n \ge 3$ and each $0 \le i \le (n-2)!$, using incomparable elements at distance $n-1$, one can obtain a shortened universal cycle for S_n of length $n! - i(n-1)$.*

Proof of Theorem 2. If $n = 3$, then possible universal cycles are for example 1232 and 145243 for $i = 1$ and $i = 0$, respectively. Let $n \ge 4$ and $0 \le i \le (n-2)!$. Let G be the cluster graph for n-permutations. By Lemma 3, G contains $(n-1)!/(n-1) = (n-2)!$ disjoint cycles formed by double edges. Let C_i be a union of i of these cycles. Let $\mathcal{P}'$ be the set satisfying $\mathcal{P} \subseteq \mathcal{P}' \subseteq \mathcal{P}^\star$, where each permutation in $\mathcal{P}^\star \setminus \mathcal{P}$ is included in $\mathcal{P}'$ if and only if it is a twin in C_i. Let G_i be the compressed cluster graph for the set of twins in C_i. Notice that G_i has $n! - i(n-1)$ edges and is still Eulerian since cycles of double edges were replaced by cycles of single edges. Let T be a tour in G_i corresponding to all of the permutations in $\mathcal{P}'$. Let G'_i be obtained from G_i by removing T. Then G'_i is balanced because G_i and T are balanced, and G'_i is strongly connected as Lemma 5 implies G'_i has a strongly connected spanning subgraph, so G'_i is Eulerian. Hence, using Lemma 6

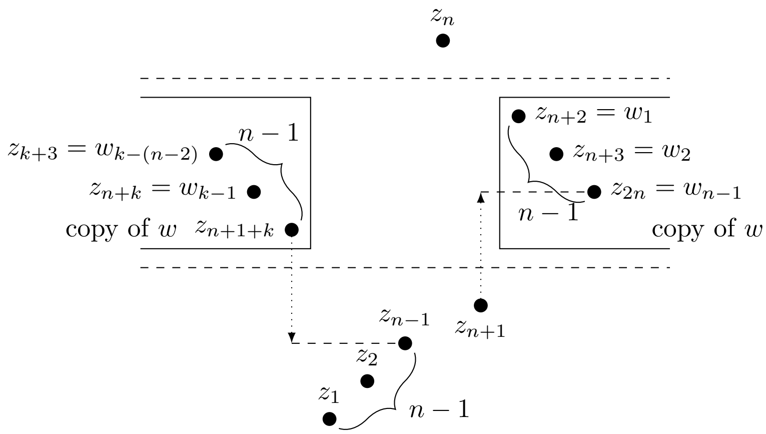


Figure 6: The word z from Lemma 7 for $n = 4$ depicted. Dashed lines indicate relative order of entries. Arrows indicate the "otherwise" cases for z_{n+1+k} and z_{n+1}.

with an Eulerian trail of G_i' that starts and ends at $(n-1, n-2, \ldots, 1)$, there exists a word $w = w_1 w_2 \cdots w_k$ whose n-windows are $L(e)$ for the edges e of G_i' in the order of the Eulerian trail, and which in particular has $\text{red}(w_1 w_2 \cdots w_{n-1}) = \text{red}(w_{k-n+2} \cdots w_k) = (n-1, n-2, \ldots, 1)$. By Lemma 7, there exists a cyclic word z, which covers each n-permutation exactly once, and covers the twins in C_i in a compressed way. This cyclic word z is a shortened universal cycle for S_n of length $n! - i(n-1)$. □

3 Conclusion

In this note we proved a conjecture of Kitaev, Potapov, and Vajnovszki [13] on shortening universal words by adding repeated elements. Our construction does not control how many different entries are in the resulting universal cycle. It would be interesting to investigate the smallest number of symbols needed to create a shortened universal cycle of a given length. As suggested in [7], it would also be interesting to determine a greedy algorithm for constructing shortened universal cycles for S_n.

The case $i = 0$ in Theorem 2 gives a new way to construct universal cycles for S_n. It is an open question whether every Eulerian tour of the cluster graph for n-permutations corresponds to a universal cycle for S_n; see [4, 11]. Our proof shows that the Eulerian tours containing the specific tour for $\mathcal{P}$ all correspond to universal cycles for S_n.

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