

The Planar Slope Number of Planar Partial 3-Trees of Bounded Degree

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Abstract. It is known that every planar graph has a planar embedding where edges are represented by non-crossing straight-line segments. We study the planar slope number, i.e., the minimum number of distinct edge-slopes in such a drawing of a planar graph with maximum degree Δ . We show that the planar slope number of every series-parallel graph of maximum degree three is three. We also show that the planar slope number of every planar partial 3-tree and also every plane partial 3-tree is at most $2^{\mathcal{O}(\Delta)}$. In particular, we answer the question of Dujmović et al. [Computational Geometry 38 (3), pp. 194–212 (2007)] whether there is a function f such that plane maximal outerplanar graphs can be drawn using at most $f(\Delta)$ slopes.

Keywords: graph drawing; planar graphs; slopes; planar slope number.

1 Introduction

The *slope number* of a graph G was introduced by Wade and Chu [10]. It is defined as the minimum number of distinct edge-slopes in a straight-line drawing of G . Clearly, the slope number of G is at most the number of edges of G , and it is at least half of the maximum degree Δ of G .

Dujmović et al. [2] asked whether there was a function f such that each graph with maximum degree Δ could be drawn using at most $f(\Delta)$ slopes. In general, the answer is *no* due to a result of Barát et al. [1]. Later, Pach and Pálvölgyi [9] and Dujmović et al. [3] proved that for every $\Delta \geq 5$, there are graphs of maximum degree Δ that need an arbitrarily large number of slopes.

On the other hand, Keszegh et al. [6] proved that every subcubic graph with at least one vertex of degree less than three can be drawn using at most four slopes; Mukkamala and Szegedy [8] extended this bound to every cubic graph. Dujmović et al. [3] give a number of bounds in terms of the maximum degree: for interval graphs, cocomparability graphs, or AT-free graphs. All the results mentioned so far are related to straight-line drawings which are not necessarily non-crossing.

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It is known that every planar graph G can be drawn so that edges of G are represented by non-crossing segments [5]. Hence, it is natural to examine the minimum number of slopes in a planar embedding of a planar graph.

In this paper, we make the (standard) distinction between *planar graphs*, which are graphs that admit a plane embedding, and *plane graphs*, which are graphs accompanied with a fixed prescribed combinatorial embedding, including a prescribed outer face. Accordingly, we distinguish between the *planar slope number* of a planar graph G , which is the smallest number of slopes needed to construct any straight-line plane embedding of G , as opposed to the *plane slope number* of a plane graph G , which is the smallest number of slopes needed to realize the prescribed combinatorial embedding of G as a straight-line plane embedding.

The research of slope parameters related to plane embedding was initiated by Dujmović et al. [2]. In [4], there are numerous results for the plane slope number of various classes of graphs. For instance, it is proved that every plane 3-tree can be drawn using at most $2n$ slopes, where n is its number of vertices. It is also shown that every 3-connected plane cubic graph can be drawn using three slopes, except for the three edges on the outer face.

In this paper, we study both the plane slope number and the planar slope number. The lower bounds of [1,9,3] for bounded-degree graphs do not apply to our case, because the constructed graphs with large slope numbers are not planar. Moreover, the upper bounds of [6,8] give drawings that contain crossings even for planar graphs.

For a fixed $k \in \mathbb{N}$, a k -tree is defined recursively as follows. A complete graph on k vertices is a k -tree. If G is a k -tree and K is a k -clique of G , then the graph formed by adding a new vertex to G and making it adjacent to all vertices of K is also a k -tree. A subgraph of a k -tree is called a *partial k -tree*.

A *two-terminal graph* (G, s, t) is a graph together with two distinct prescribed vertices $s, t \in V(G)$, known as *terminals*. The vertex s is called *source* and t is called *sink*. For a pair $(G_1, s_1, t_1), (G_2, s_2, t_2)$ of two-terminal graphs, a *serialization* is an operation that identifies t_1 with s_2 , yielding a new two-terminal graph with terminals s_1 and t_2 . Similarly, a *parallelization* is an operation which consists of identifying s_1 with s_2 into a single vertex s , and t_1 with t_2 into a single vertex t , thus yielding a two-terminal graph with terminals s and t . A two-terminal graph (G, s, t) is called *series-parallel graph* or *SP-graph* for short, if it either consists of a single edge connecting the vertices s and t , or if it can be obtained from smaller SP-graphs by serialization or parallelization.

We present several upper bounds on the plane and planar slope number in terms of the maximum degree Δ . The most general result of this paper is the following theorem, which deals with plane partial 3-trees.

Theorem 1. *The plane slope number of any plane partial 3-tree with maximum degree Δ is at most $2^{\mathcal{O}(\Delta)}$.*

Note that the above theorem implies that the planar slope number of any partial planar 3-tree is also at most $2^{\mathcal{O}(\Delta)}$. Since every outerplanar graph is also a partial 3-tree, the result above answers a question of Dujmović et al. [4], who asked

whether a plane maximal outerplanar graph can be drawn using at most $f(\Delta)$ slopes.

In this extended abstract, we omit the proof of Theorem 1. Section 3 contains the proof of a weaker version of this result which deals with (non-partial) plane 3-trees.

In the special case of series-parallel graphs of maximum degree at most 3, we are able to prove an even better (in fact optimal) upper bound.

Theorem 2. *Any series-parallel graph with maximum degree at most 3 has planar slope number at most 3.*

Parts of the proof of Theorem 2 are in Section 2.

Let us introduce some basic terminology and notation that will be used throughout this paper. Let s be a segment in the plane. The smallest angle $\alpha \in [0, \pi)$ such that s can be made horizontal by a clockwise rotation by α , is called the *slope of s* . The *directed slope* of a directed segment is an angle $\alpha' \in [0, 2\pi)$ defined analogously.

A plane graph is called a *near triangulation* if all faces, except the outer face, are triangles.

2 Series-Parallel Graphs

In this section, we show the main ideas of the proof of Theorem 2.

We will in fact show that any series-parallel graph of maximum degree three can be embedded using the slopes from the set $S = \{0, \pi/4, -\pi/4\}$. This particular choice of S is purely aesthetic. Throughout this section, segments of slope $\pi/4$ (or 0, or $-\pi/4$) will be known as *increasing* (or *horizontal*, or *decreasing*, respectively).

First we give some useful definitions. For a pair of integers j and k , we say that a series-parallel graph (G, s, t) is a (j, k) -graph if G has maximum degree three, and furthermore, the vertex s has degree at most j and the vertex t has degree at most k .

Let us begin by a simple but useful lemma whose proof is omitted.

Lemma 1. *Let (G, s, t) be a $(1, 1)$ -graph. Then G is either a single edge, a serialization of two edges, or a serialization of three graphs G_1 , G_2 and G_3 , where G_1 and G_3 consist of a single edge and G_2 is a $(2, 2)$ -graph.*

We proceed with more terminology. An *up-triangle* abc is a right isosceles triangle whose hypotenuse ab is horizontal and whose vertex c is above the hypotenuse. We say that a series parallel graph (G, s, t) has an *up-triangle embedding* if it can be embedded inside an up-triangle abc using the slopes from S , in such a way that the two vertices s and t coincide with the two endpoints of the hypotenuse of abc , and all the remaining vertices are either inside or on the boundary of abc .

The concept of up-triangle embedding is motivated by the following lemma.

Lemma 2. *Every $(2, 2)$ -graph has an up-triangle embedding.*

Proof. Let (G, s, t) be a $(2, 2)$ -graph. We proceed by induction on the size of G . If G is a single edge, it obviously has an up-triangle embedding. If G is obtained by serialization or parallelization then there are a few cases to discuss. They are depicted in Fig. 1. \square

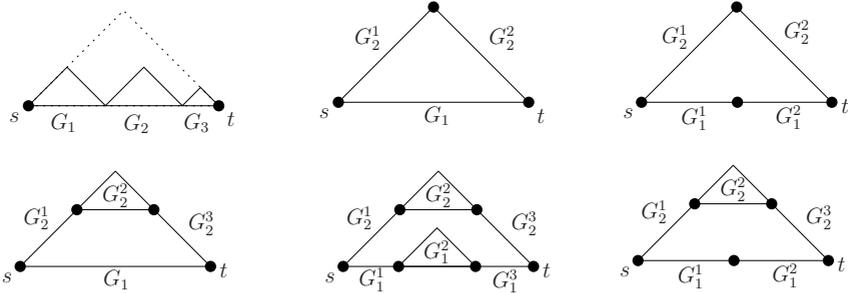


Fig. 1. Possible construction of a $(2, 2)$ -graph G by serialization and parallelization of $(1, 1)$ -graphs

To deal with $(3, 2)$ -graphs, we need a more general concept than up-triangle embeddings. To this end, we introduce the following definitions.

An *up-spade* is a convex pentagon with vertices a, b, c, d, e in counterclockwise order, such that the segment ab is decreasing, the segment bc is horizontal, the segment cd is increasing, the segment ed is decreasing and the segment ae is increasing. We say that a series-parallel graph (G, s, t) has an *up-spade embedding* if it can be embedded into an up-spade $abcde$ using the slopes from S , in such a way that the vertex s coincides with the point a , the vertex t coincides either with the point b or with the point c , and all the remaining vertices of G are inside or on the boundary of the up-spade. Analogously, a *reverse up-spade embedding* is an embedding of a series-parallel graph (G, s, t) in which s coincides with b or c and t coincides with d .

Lemma 3. *Every $(3, 2)$ -graph (G, s, t) has an up-spade embedding or an up-triangle embedding. Similarly, every $(2, 3)$ -graph (G, s, t) has a reverse up-spade embedding or an up-triangle embedding.*

Proof. It suffices to prove just the first part of the lemma; the other part is symmetric. We again proceed by induction.

Let (G, s, t) be a $(3, 2)$ -graph. If G is also a $(2, 2)$ -graph, then G has an up-triangle embedding by Lemma 2. Assume that G is not a $(2, 2)$ -graph. It is easy to see that in such case G has no up-triangle embedding, since it is impossible to embed three edges into an up-triangle in such a way that they meet in the endpoint of its hypotenuse.

Assume that G has been obtained by a serialization of a sequence of graphs G_1, G_2, \dots, G_k , and that each of the graphs G_i is a single edge or a parallelization

of smaller graphs. It follows that the graph G_2 is a single edge, because otherwise the two graphs G_1 and G_2 would share a vertex of degree at least 4. Let G_3^+ be the (possibly empty) serialization of G_3, \dots, G_k . If G_3^+ is nonempty, it has an up-triangle embedding by Lemma 2. The graph G_1 has an up-spade embedding by induction. We may combine these embeddings as shown in Fig. 2 to obtain an up-spade embedding of G . If G_3^+ is empty, the construction is even simpler.

Assume now that G has been obtained by parallelization. Necessarily, it was a parallelization of a $(1, 1)$ -graph G_1 and a $(2, 1)$ -graph G_2 . The graph G_2 can then be obtained by a serialization of a $(2, 2)$ -graph G_2^1 and a single edge G_2^2 . The graph G_2^1 has an up-triangle embedding. Combining these embeddings, we obtain an up-spade embedding of G , as shown in Fig. 2. Note that we distinguish the possible structure of G_1 using Lemma 1. □

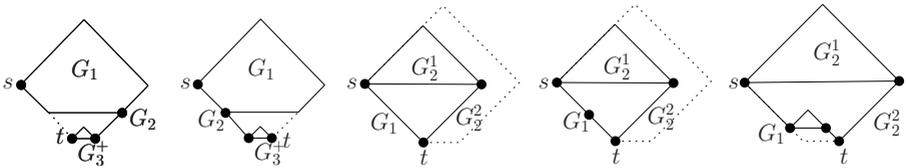


Fig. 2. Constructing an up-spade embedding of a $(3, 2)$ -graph by serialization and parallelization of smaller graphs

A similar case analysis can be done also for a $(3, 3)$ -graph. The serialization is easy by connecting two $(3, 2)$ -graphs while the parallelization takes a few cases. This finishes the proof of Theorem 2.

3 Planar 3-Trees

In this section, we outline a proof of a considerably weaker version of Theorem 1. Our current goal is to prove the following result.

Theorem 3. *There is a function g , such that every plane 3-tree with maximum degree Δ can be drawn using at most $g(\Delta)$ slopes.*

It is known that any plane 3-tree can be generated from a triangle by a sequence of vertex-insertions into inner faces. Here, a *vertex-insertion* is an operation that consists of creating a new vertex in the interior of a face, and then connecting the new vertex to all the three vertices of the surrounding face, thus subdividing the face into three new faces.

Throughout this section, we assume that Δ is a fixed integer.

For a partial plane 3-tree G we define the *level* of a vertex v as the smallest integer k such there is a set V_0 of k vertices of G with the property that v is on the outer face of the plane graph $G - V_0$. Let G be a partial plane 3-tree. An edge of G is called *balanced* if it connects two vertices of the same level of G . An

edge that is not balanced is called *tilted*. Similarly, a face of G whose vertices all belong to the same level is called balanced, and any other face is called tilted. In a 3-tree, the level of a vertex v can also be equivalently defined as the length of the shortest path from v to a vertex on the outer face. However, this definition cannot be used for plane partial 3-trees.

Note that whenever we insert a new vertex v into an inner face of a 3-tree, the level of v is one higher than the minimum level of its three neighbors; note also that the level of all the remaining vertices of the 3-tree is not affected by the insertion of a new vertex.

Recall that a near triangulation is a plane graph whose every inner face is a triangle.

Let u, v be a pair of vertices forming an edge. A *bubble* over uv is an outerplanar plane near triangulation that contains the edge uv on the boundary of the outer face. The edge uv is called the *root* of the bubble. An *empty bubble* is a bubble that has no other edge apart from the root edge. A *double bubble* over uv is a union of two bubbles over uv which have only u and v in common and are attached to uv from its opposite sides. A *leg* is a graph L created from a path P by adding a double bubble over every edge of P . The path P is called the *spine of L* and the endpoints of P are also referred to as the endpoints of the leg. Note that a single vertex is also considered to form a leg.

A *tripod* is a union of three legs which share a common endpoint. A *spine* of a tripod is the union of the spines of its legs. Observe that a tripod is an outerplanar graph. The vertex that is shared by all the three legs of a tripod is called the *central vertex*.

Let G be a near triangulation, let Φ be an inner face of G . Let T be a tripod with three legs X, Y, Z and a central vertex c . An *insertion of tripod T into the face Φ* is the operation performed as follows. First, insert the central vertex c into the interior of Φ and connect it by edges to the three vertices of Φ . This subdivides Φ into three subfaces. Extend c into an embedding of the whole tripod T , by embedding a single leg of the tripod into the interior of each of the three subfaces. Next, connect every non-central vertex of the spine of the tripod to the two vertices of Φ that share a face with the corresponding leg. Finally, connect each non-spine vertex v of the tripod to the single vertex of Φ that shares a face with v . See Fig. 3. Observe that the graph obtained by a tripod insertion into Φ is again a near triangulation.

Lemma 4. *Let G be a graph. The following statements are equivalent:*

1. G is a plane 3-tree, i.e., G can be created from a triangle by a sequence of vertex insertions into inner faces.
2. G can be created from a triangle by a sequence of tripod insertions into inner faces.
3. G can be created from a triangle by a sequence of tripod insertions into balanced inner faces.

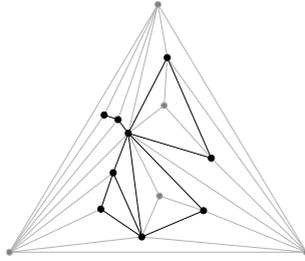


Fig. 3. An example of a tripod consisting of vertices of level 1 in a plane 3-tree

Proof. Clearly, (3) implies (2).

To observe that (2) implies (1), it suffices to notice that a tripod insertion into a face Φ can be simulated by a sequence of vertex insertions: first insert the central vertex of a tripod into Φ , then insert the vertices of the spine into the resulting subfaces, and then create each bubble by inserting vertices into the face that contains the root of the bubble and its subsequent subfaces.

To show that (1) implies (3), proceed by induction on the number of levels in G . If G only has vertices of level 0, then it consists of a single triangle and there is nothing to prove. Assume now that the G is a graph that contains vertices of $k > 0$ distinct levels, and assume that any 3-tree with fewer levels can be generated by a sequence of balanced tripod insertions by induction.

We will show that the vertices of level exactly k induce in G a subgraph whose every connected component is a tripod, and that each of these tripods is inserted inside a triangle whose vertices have level $k - 1$.

Let C be a connected component of the subgraph induced in G by the vertices of level k . Let v_1, v_2, \dots, v_m be the vertices of C , in the order in which they were inserted when G was created by a sequence of vertex insertions. Let Φ be the triangle into which the vertex v_1 was inserted, and let x, y and z be the vertices of Φ . Necessarily, all three of these vertices have level $k - 1$. Each of the vertices v_2, \dots, v_m must have been inserted into the interior of Φ , and each of them must have been inserted into a face that contained at least one of the three vertices of Φ .

Note that at each point after the insertion of v_1 , there are exactly three faces inside Φ that contain a pair of vertices of Φ ; each of these three faces is incident to an edge of Φ . Whenever a vertex v_i is inserted into such a face, the subgraph induced by vertices of level k grows by a single edge. These edges form a union of three paths that share the vertex v_1 as their common endpoint.

On the other hand, when a vertex v_i is inserted into a face formed by a single vertex of Φ and a pair of previously inserted vertices v_j, v_ℓ , then the graph induced by vertices of level k grows by two edges forming a triangular face with another edge whose endpoints have level k .

With these observations, it is easily checked (e.g., by induction on i) that for every $i \geq 1$, the subgraph of G induced by the vertices v_1, \dots, v_i is a tripod inserted into Φ . From this fact, it follows that the whole graph G could be created by a sequence of tripod insertions into balanced faces. □

Note that when we insert a tripod into a balanced face, all the vertices of the tripod will have the same level (which will be one higher than the level of the face into which we insert the tripod). In particular, each balanced face we create by this insertion is an inner face of the tripod that we insert.

We will use the construction of plane 3-trees by tripod insertions as a main tool of our proof. Note that if G is a plane 3-tree of maximum degree at most Δ , then any tripod T used in the construction of G has fewer than 3Δ vertices. This is because every vertex of T is adjacent to a vertex of the triangular face Φ into which T was inserted, but each vertex of Φ has fewer than Δ neighbors on T . Let us say that a tripod T is Δ -bounded if it has maximum degree at most Δ and if it has at most 3Δ vertices. We conclude that any plane 3-tree of maximum degree Δ can be constructed by insertions of Δ -bounded tripods into balanced inner faces.

Let us give some technical definitions. Let α be a directed slope and let p be a point. We use the notation (p, α) to denote the ray starting in p with direction α .

Let G be a plane graph, let v be a vertex of G . We say that the vertex v has *visibility in direction α with respect to G* , if the ray starting in v and having direction α does not intersect the embedding of G in any point except v .

Assume now that G is a graph that has been obtained by inserting a tripod T in to a triangle Φ with vertex set x, y, z . Assume that we are given an embedding of the three vertices x, y, z as points in the plane, and we are also given a plane embedding \mathcal{E}_T of the tripod T . We say that the embedding \mathcal{E}_T is *compatible* with the embedding of x, y, z , if \mathcal{E}_T is inside the convex hull of x, y, z , and it is possible to extend the plane embedding $\mathcal{E}_T \cup \{x, y, z\}$ into a plane straight-line embedding of the whole graph G .

Let us explain in more detail the main idea of the proof. As the principal step, we show that for every tripod T with at most 3Δ vertices, there is a finite set \mathcal{F}_T of “permissible” embeddings of T , with the property that for any triangle x, y, z embedded in the plane, there exists an embedding from \mathcal{F}_T whose appropriately scaled and translated copy is compatible with x, y, z . Since there are only finitely many tripods to consider, and since each considered tripod has only finitely many embeddings specified, all these embeddings together only define finitely many slopes, and finitely many (up to scaling) distinct triangular faces.

We thus have only finitely many pairs (Φ, T) , where Φ is an embedding of a triangular face appearing in a permissible embedding of a tripod T' , and T is a tripod. For each of these pairs we select a permissible embedding \mathcal{E}_T of the tripod T that is compatible with Φ . Whenever we want to insert T into a scaled copy of the face Φ , we use the appropriately scaled copy of \mathcal{E}_T , so that the slope of a segment connecting a given vertex of Φ to a given vertex of \mathcal{E}_T will only depend on the two vertices but not on the scaling of Φ .

As we know, any plane 3-tree G can be constructed as a sequence of tripod insertions into balanced faces. We construct the embedding of G recursively, so that whenever we need to insert a tripod T into an already embedded balanced triangle, we use the embedding selected by the procedure from the previous

paragraph. The total number of slopes of all the balanced edges in the embedding of G can then be bounded by the total number of slopes appearing in the permissible embeddings of all tripods. The total number of slopes of tilted edges is bounded as well, which follows from the argument at the end of the last paragraph.

Let us now turn towards the technical details of the argument.

Lemma 5. *Let uv be a horizontal segment in the plane, let H be a halfplane containing uv on its boundary and extending above uv , and let $\varphi \in (0, \pi/2)$ be an angle. Let z be the point in H such that the segments uz and vz have slopes φ and $-\varphi$, respectively. There is a set $S \subseteq (-\varphi, \varphi)$ of 2Δ slopes such that every bubble B with root uv has a straight line drawing using only the slopes from S . Furthermore, all the vertices of this drawing except u and v are in the interior of the triangle uvz , and each vertex has visibility in any direction $\alpha \in (\varphi, \pi - \varphi)$.*

Proof (Sketch of proof of Lemma 5). Assume φ and B are given. To construct the drawing, first fix a sequence of slopes $0 < \varphi_0 < \varphi_1 < \varphi_2 < \dots < \varphi_{\Delta-2} < \varphi$. In the first step, draw the vertices adjacent to u or v on a common line parallel to line uv , such that the absolute values of the slopes of the edges between uv and their neighbors belong to the sequence $\varphi_0, \dots, \varphi_{\Delta-2}$ (see Fig. 4).

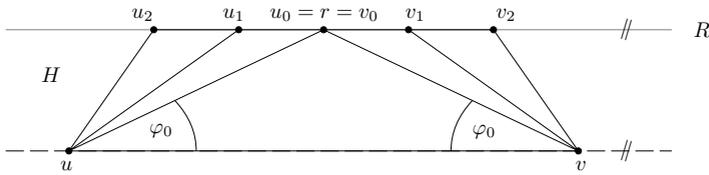


Fig. 4. Illustration of the proof of Lemma 5: drawing vertices adjacent to u and v

The rest of the bubble B can be expressed as a union of smaller bubbles, each of them rooted at a horizontal edge that has been drawn in the first step. We recursively apply the same drawing procedure to draw each of these smaller bubbles, each of them inside its own triangle similar to uvz , as illustrated in Fig. 5. □

Now that we can draw isolated bubbles, we may describe how to combine these drawings into a drawing of the whole leg of a tripod. Simply speaking, the procedure concatenates the drawings from Lemma 5 (appropriately rotated) on a single prescribed ray R .

Leg Drawing Procedure (LDP):

Input: A leg L with the central vertex u already drawn. A ray R with origin in u .

Output: Drawing of the leg L .

1. Assume that the spine of leg L contains vertices $u = u_0, u_1, \dots, u_k$ such that $u_i u_{i-1}$ is an edge for $0 < i \leq k$.

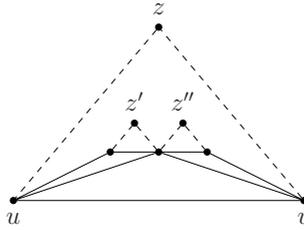


Fig. 5. Drawing a bubble in a triangle

2. Draw vertices u_i on R such that $|u_{i-1} - u| < |u_i - u|$ for $0 < i \leq k$.
3. Fix an angle $\varphi \in (0, \pi/2)$.
4. Use the drawing from Lemma 5, rotated and reflected if necessary, on both bubbles rooted at the edge $u_{i-1}u_i$ for $0 < i \leq k$. All the drawings use the same value of φ , and hence the same set S of slopes.

It is again not difficult to check that this procedure generates a correct plane straight-line drawing of a given leg. A careful analysis allows us to conclude that the drawing will use at most 2Δ slopes, and contain at most 4Δ distinct triangular faces, up to scaling and translation. These slopes and face-types only depend on the slope of the ray R and the choice of φ . Since by the choice of φ we can force each bubble to be embedded inside an arbitrarily “flat” isosceles triangle, we can easily argue that any vertex of the spine has visibility in any direction that differs from the undirected slope of R by more than φ . Moreover, a vertex $v \in L$ that does not belong to the spine has visibility in those directions that differ from the slope of R by more than φ and are directed towards the half-plane of R containing v . Finally, the central vertex u has visibility in any direction that differs from the directed slope of R by more than φ .

Finally we describe a procedure that combines the drawing of the individual legs into the drawing of the whole tripod. Let ε denote the value $\frac{\pi}{100}$ (any sufficiently small integral fraction of π is suitable here). The procedure expects a triangle Φ whose vertices are three points a, b, c . It then selects the position of a central vertex u , as well as the slopes of three rays R_1, R_2, R_3 emanating from u , and then draws the three legs of a given tripod on these rays by using LDP. The slopes of the legs are chosen in such a way that the resulting embedding of the tripod is compatible with Φ .

Furthermore, the slopes of the three rays are rounded to an integral multiple of ε . This rounding ensures, that the slopes of the spines of the legs can only take finitely many values (namely, at most $\frac{2\pi}{\varepsilon}$). It will follow that the procedure can only generate (up to scaling) a finite number of tripod embeddings, for all possible triangles Φ .

Tripod Drawing Procedure (TDP):

Input: A triangle $\Phi = \{a, b, c\}$ and a tripod T , where Φ is already drawn.
 Output: Drawing of all vertices and edges of the tripod T inside Φ .

1. Fix an angle $\varphi \in (0, \frac{\pi}{4} - \varepsilon)$
2. Let u be the central vertex of T and let L_i for $i \in \{1, 2, 3\}$ be the legs of T .
3. Draw u to the intersection of the axes of the inner angles of Φ .
4. Process leg L_i for $i \in \{1, 2, 3\}$:
 - (a) Let e be the endpoint of the spine of L_i different from u .
 - (b) Let x, y be the vertices of Φ adjacent to the end e of L_i .
 - (c) Let o be the axis of the angle cud .
 - (d) Let R_i be a ray originating at u of slope o rounded to integral multiple of ε .
 - (e) Use LDP to draw L_i on R_i . Scale the result so that it fits inside Ψ .

It is not difficult to check that the tripod-drawing procedure produces a correct straight-line embedding of any tripod inside any triangle Φ . Moreover, there is a set of slopes S of size at most $4\Delta\pi\varepsilon^{-1}$ and a set of triangles τ of size at most $8\Delta\pi\varepsilon^{-1}$, such that for any tripod T and any triangle Φ , the resulting embedding of T only uses the slopes from the set S and all its inner faces are scaled copies of triangles from τ .

The visibility properties and the ‘‘flatness’’ of the leg embedding guarantee that the resulting tripod embedding is compatible with Φ .

Let us now consider the slopes of the ‘tilted’ segments, i.e., those segments that connect a vertex of Φ with a vertex of the tripod embedded inside Φ by TDP. Assume that Φ is fixed. For each Δ -bounded tripod T , there are at most 9Δ segments connecting a vertex of Φ with a vertex of T . The number of Δ -bounded tripods is clearly finite (in fact, an upper bound of the form $2^{\mathcal{O}(\Delta)}$ can be obtained without much difficulty). We may now easily see that, for a fixed Φ , the total number of slopes of the segments that connect a vertex of Φ with a vertex of a Δ -bounded tripod is bounded.

Of course, for different triangles Φ , different slopes of this type arise. However, this is not an issue for us, because to generate a plane embedding of a plane 3-tree of maximum degree at most Δ , it is sufficient to insert Δ -bounded tripods into faces of previously inserted tripods. Thus, there are only finitely many triangles Φ for which we ever need to perform the tripod drawing procedure. Thus, by repeated calls of TDP, we may construct an embedding of any plane 3-tree with maximum degree Δ , while using at most $g(\Delta)$ slopes. More careful analysis of these arguments reveals a bound of the form $g(\Delta) = 2^{\mathcal{O}(\Delta)}$.

4 Conclusion and Open Problems

We have presented an upper bound of $2^{\mathcal{O}(\Delta)}$ for the planar slope number of planar partial 3-trees of maximum degree Δ . It is not obvious to us if the used methods can be generalized to a larger class of graphs, such as planar partial k -trees of bounded degree.

Let us remark that our proof of Theorem 1 actually implies a slightly stronger statement: for any Δ there is a set of slopes $S = S(\Delta)$ of size $2^{\mathcal{O}(\Delta)}$, such that all partial plane 3-trees of maximum degree Δ can be drawn using the slopes of S . This implies, for instance, that there is a constant $\varepsilon = \varepsilon(\Delta) > 0$ such that

in our drawing of a partial plane 3-tree of maximum degree Δ , any two edges sharing a vertex have slopes differing by at least ε . Our method, however, is not necessarily suitable for obtaining good bounds on ε .

In view of the results of Keszegh et al. [6] and Mukkamala and Szegedy [8] for the slope number of (sub)cubic planar graphs, it would also be interesting to find analogous bounds for the planar slope number.

The main open problem is to determine whether the planar slope number of a planar graph can be bounded from above by a function of its maximum degree.

This paper does not address lower bounds for the planar slope number in terms of Δ ; this might be another direction worth pursuing.

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